Differential Equations

Text: 12.3, 12.5 Lecture Notes: 13-15 Lecture Slides: 6

Quote of Ad Hoc Homework One Solutions	
	Ween : Roses are Free (1994)

1. ODE Review

When solving the linear wave equation, heat equation and Poisson's partial differential equation (PDE), on compact domains of $\mathbb{R}^n$ separation of variables is the typical method. When using separation of variables one trades a PDE for a class of ordinary differential equations (ODE) that manifest a set of orthogonal functions that can be used to represent the solution to the original PDE. For most of our work we will concentrate on,

$$y'' + \lambda y = 0, \ \lambda \in \mathbb{R}, \ x \in (0, L). \tag{1}$$

1.1. **General Solution to the ODE.** Justify that the following functions solve the previous ODE for particular values of λ and arbitrary constants $c_i \in \mathbb{R}$ for $i = 1, 2, 3, 4, 5, 6$.[1]

Case	Function 1	Function 2
$\lambda > 0$	$y(x) = c_1 \cos(\sqrt{\lambda}x)$	$y(x) = c_2 \sin(\sqrt{\lambda}x)$
$\lambda < 0$	$y(x) = c_3 \cosh(\sqrt{\lvert\lambda\rvert}x)$	$y(x) = c_4 \sinh(\sqrt{\lvert\lambda\rvert}x)$
$\lambda = 0$	$y(x) = c_5$	$y(x) = c_6 x$

To solve this problem we make the assumption that $y(x) = e^{rx}$ to get that,

$$y'' + \lambda y = r^2 e^{rx} + \lambda e^{rx} = e^{rx}\left(r^2 + \lambda\right) = 0, \tag{2}$$

which has roots $r = \mp\sqrt{-\lambda}$. This leads to the following general solutions, which depend on the value of λ,

$$\lambda > 0 : y_1(x) = b_1 e^{i\sqrt{\lambda}x} + b_2 e^{-i\sqrt{\lambda}x}, \tag{3}$$

$$\lambda < 0 : y_2(x) = b_3 e^{\sqrt{\lvert\lambda\rvert}x} + b_4 e^{-\sqrt{\lvert\lambda\rvert}x}, \tag{4}$$

$$\lambda = 0 : y_3(x) = b_5 e^{0\cdot x} + b_6 x e^{0\cdot x}. \tag{5}$$

Often it is easier to express these solutions in terms of the *trigonometric* and *hyperbolic trigonometric* functions. To do this we note the following,

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}, \tag{6}$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}, \tag{7}$$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2}, \tag{8}$$

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2}. \tag{9}$$

[1] There are, of course, many ways to do this. You could re-derive the given information, quote a previous homework or substitute the solutions into the ODE. You choice.

This allows us to rewrite the previous solutions in terms of the above functions. [2] Doing so gives,

$$\lambda > 0 : y_1(x) = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x), \tag{14}$$

$$\lambda < 0 : y_2(x) = c_3 \cosh\left(\sqrt{|\lambda|} x\right) + c_4 \sinh\left(\sqrt{|\lambda|} x\right), \tag{15}$$

$$\lambda = 0 : y_3(x) = b_5 + b_6 x. \tag{16}$$

These are the general solutions to (**??**) based on the value of λ.

2. BVP Overview

Boundary value problems (BVP) typically arise within the context of PDE, which are equations modelling the evolution of a quantity in both space and time. There are important general results for BVP, which are set within the context of Sturm-Liouville problems.[3] What can be efficiently done by hand tends to be limited. The problem, in Cartesian coordinates, is to find all solutions to,

$$y'' + \lambda y = 0, \ \lambda \in \mathbb{R}, \ x \in (0, L), \tag{17}$$

which also satisfy,

$$l_1 y(0) + l_2 y'(0) = 0, \tag{18}$$

$$r_1 y(L) + r_2 y'(L) = 0. \tag{19}$$

This problem is intractable, by hand, for general values of l_1, l_2, r_1, r_2. However, the following set of values,

	l_1	l_2	r_1	r_2
Case I	1	0	1	0
Case II	0	1	0	1
Case III	1	0	0	1
Case IV	0	1	1	0

lead to BVP that can be solved by hand.

2.1. Application of Boundary Conditions.

We have seen Case I in long homework 2 and Case II in long homework 3. Now we concentrate on Case III and Case IV. Before you begin you may want to collect the previous results so that you have them all on one page.

2.1.1. *Case III.*

From the previous table of functions, first show that $y(0) = 0$ implies that $c_1 = c_3 = c_5 = 0$. Next show that $y'(L) = 0$ implies that $c_4 = c_6 = 0$. This leaves just the sine function to deal with. Lastly, show that $y(x) = c_2 \sin(\sqrt{\lambda} x)$ satisfies the condition $y'(L) = 0$ for the specific values $\sqrt{\lambda} = (2n+1)\frac{\pi}{2L}$ where $n = 1, 2, 3, \ldots$.

We need to find which of the six functions non-trivially satisfy $y(0) = 0$. This implies that we need to keep functions passing through the point $(0, 0)$. Of the six listed only the functions from 'Function 2' column do this. So, we know that $c_1 = c_3 = c_5 = 0$. Now if we apply the second boundary condition, $y'(L) = 0$, which implies that when $x = L$ the function must have a horizontal tangent line. Immediately, this implies that $c_6 = 0$ since c_6 controls the slope of the line. Next we have the following,

$$y'(L) = c_4 \sqrt{|\lambda|} \cosh(\sqrt{|\lambda|} L) \tag{20}$$

$$= c_4 \sqrt{|\lambda|} \frac{e^{\sqrt{|\lambda|} L} + e^{-\sqrt{|\lambda|} L}}{2} = 0. \tag{21}$$

Since the exponential functions are never negative the cosh function is never zero. Also, since $\lambda \neq 0$ we must conclude that $c_4 = 0$. For the final function we have that,

$$y'(L) = c_2 \sqrt{\lambda} \cos(\sqrt{\lambda} L) = 0, \tag{22}$$

[2] For example,

$$y_1(x) = b_1 e^{i\sqrt{\lambda} x} + b_2 e^{-i\sqrt{\lambda} x} \tag{10}$$

$$= \frac{c_1 - ic_2}{2} e^{i\sqrt{\lambda} x} + \frac{c_1 + ic_2}{2} e^{-i\sqrt{\lambda} x} \tag{11}$$

$$= c_1 \left[\frac{e^{i\sqrt{\lambda} x} + e^{-i\sqrt{\lambda} x}}{2}\right] + c_2 \left[\frac{e^{i\sqrt{\lambda} x} - e^{-i\sqrt{\lambda} x}}{2i}\right] \tag{12}$$

$$= c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x). \tag{13}$$

The hyperbolic trigonometric functions have a similar derivation.

[3] Some of these results can be found in long homework number 7

which can be true for $\sqrt{\lambda}L = (2n-1)\frac{\pi}{2}$.

2.1.2. *Case IV.* From the previous table of functions, first show that $y'(0) = 0$ implies that $c_2 = c_4 = c_6 = 0$. Next show that $y(L) = 0$ implies that $c_3 = c_5 = 0$. This leaves just the cosine function to deal with. Lastly, show that $y(x) = c_1\cos(\sqrt{\lambda}x)$ satisfies the condition $y(L) = 0$ for the specific values $\sqrt{\lambda} = (2n+1)\frac{\pi}{2L}$ where $n = 1, 2, 3, \ldots$.
We need to find which of the six functions non-trivially satisfy $y'(0) = 0$. This implies that we need to keep functions with a horizontal tangent when $x = 0$. Of the six listed only the functions from 'Function 1' column have this. So, we know that $c_2 = c_4 = c_6 = 0$. Now if we apply the second boundary condition, $y(L) = 0$, which implies that when $x = L$ the function must have a root. Immediately, this implies that $c_5 = 0$ since c_5 controls the horizontal offset of the line. Next we have the following,

$$y(L) = c_3\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}L) \tag{23}$$

$$= c_3\sqrt{|\lambda|}\frac{e^{\sqrt{|\lambda|}L} + e^{-\sqrt{|\lambda|}L}}{2} = 0. \tag{24}$$

Since the exponential functions are never negative the cosh function is never zero. Also, since $\lambda \neq 0$ we must conclude that $c_3 = 0$. For the final function we have that,

$$y(L) = c_1\sqrt{\lambda}\cos(\sqrt{\lambda}L) = 0, \tag{25}$$

which can be true for $\sqrt{\lambda}L = (2n-1)\frac{\pi}{2}$.

3. Power-Series Solutions to ODE's and Hyperbolic Trigonometric Functions

Consider the ordinary differential equation:

$$y'' - y = 0 \tag{26}$$

For solutions see second PDF

3.1. **General Solution - Standard Form.** Show that the solution to (26) is given by $y(x) = c_1e^x + c_2e^{-x}$.

3.2. **General Solution - Nonstandard Form.** Show that $y(x) = b_1\sinh(x) + b_2\cosh(x)$ is a solution to (26) where $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$ and $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$.

3.3. **Conversion from Standard to Nonstandard Form.** Show that if $c_1 = \dfrac{b_1 + b_2}{2}$ and $c_2 = \dfrac{b_1 - b_2}{2}$ then $y(x) = c_1e^x + c_2e^{-x} = b_1\cosh(x) + b_2\sinh(x)$.

3.4. **Relation to Power-Series.** Assume that $y(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$ to find the general solution of (26) in terms of the hyperbolic sine and cosine functions. [4]

4. Heat Equation on a closed and bounded spatial domain of $\mathbb{R}^{1+1}$

Consider the one-dimensional heat equation,

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \quad , \tag{28}$$

$$x \in (0, L)\,, \qquad t \in (0, \infty)\,, \qquad c^2 = \frac{K}{\sigma\rho}. \tag{29}$$

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[4]The hyperbolic sine and cosine have the following Taylor's series representations centred about $x = 0$,

$$\cosh(x) = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} \qquad \sinh(x) = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}. \tag{27}$$

It is worth noting that these are basically the same Taylor series as cosine/sine with the exception that the signs of the terms do not alternate. From this we can gather a final connection for wrapping all of these functions together. If you have the Taylor series for the exponential function and extract the even terms from it then you have the hyperbolic cosine function. Taking the hyperbolic cosine function and alternating the sign of its terms gives you the cosine function. Extracting the odd terms from the exponential function gives the same statements for the hyperbolic sine and sine functions. The reason these functions are connected via the imaginary number system is because when i is raised to integer powers it will produce these exact sign alternations. So, if you remember $e^x = \sum_{n=0}^{\infty} x^n/n!$ and $i = \sqrt{-1}$ then the rest (hyperbolic and non-hyperbolic trigonometric functions) follows!

Equations (28)-(29) model the time-evolution of the temperature, $u = u(x,t)$, of a heat conducting medium in one-dimension. The object, of length L, is assumed to have a homogenous thermal conductivity K, specific heat σ, and linear density ρ. That is, $K, \sigma, \rho \in \mathbb{R}^+$. If we consider an object of finite-length, positioned on say $(0, L)$, then we must also specify the boundary conditions[5],

$$u_x(0,t) = 0, u_x(L,t) = 0, . \tag{30}$$

Lastly, for the problem to admit a unique solution we must know the initial configuration of the temperature,

$$u(x,0) = f(x). \tag{31}$$

4.1. **Separation of Variables : General Solution.** Assume that the solution to (28)-(29) is such that $u(x,t) = F(x)G(t)$ and use separation of variables to find the general solution to (28)-(29), which satisfies (30)-(31). [6]

(1) Assume that, $u(x,t) = F(x)G(t)$ then $u_{xx} = F''(x)G(t)$ and $u_t = F(x)G'(t)$ and the 1-D heat equation becomes,

$$\frac{G'(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = -\lambda, \tag{32}$$

where we have introduced the separation constant λ.[7] From this equation we have the two ODE's,

$$\begin{aligned} G'(t) + \lambda c^2 G(t) &= 0, && (33)\\ F''(x) + \lambda F(x) &= 0. && (34) \end{aligned}$$

Each of these ODE's can be solved through 'elementary methods' to get,[8]

$$\begin{aligned} \lambda \in \mathbb{R} &\ :\ G(t) = Ae^{-\lambda c^2 t}, \quad A \in \mathbb{R}, && (35)\\ \lambda \in \mathbb{R}^+ &\ :\ F(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x), && (36)\\ \lambda \in \mathbb{R}^- &\ :\ F(x) = c_3 \cosh(\sqrt{|\lambda|}x) + c_4 \sinh(\sqrt{|\lambda|}x), && (37)\\ \lambda = 0 &\ :\ F(x) = c_5 + c_6 x. && (38) \end{aligned}$$

Each of the functions $F(x)$ must also satisfy the boundary conditions, $u_x(0,t) = 0$ and $u_x(L,t)$ and so we won't need all of them. Notice that the boundary conditions imply that,

$$\begin{aligned} u_x(0,t) &= F'(0)G(t) = 0, && (39)\\ u_x(L,t) &= F'(L)G(t) = 0, && (40) \end{aligned}$$

which gives $F'(0) = 0$ and $F'(L) = 0$.[9] So, we now have to determine, which of the previous functions, $F(x)$, satisfy these boundary conditions. To this end we have the following arguments,

$$\begin{aligned} \lambda \in \mathbb{R}^+ &\ :\ F'(0) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}0) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}0) = c_1\sqrt{\lambda}\cdot 0 + c_2\sqrt{\lambda}\cdot 1 \implies c_2 = 0,\\ \lambda \in \mathbb{R}^+ &\ :\ F'(L) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}L) = c_1\sqrt{\lambda}\cdot\sin(\sqrt{\lambda}L) + 0\cdot\sqrt{\lambda}\cos(\sqrt{\lambda}L) \implies\\ &\implies c_1\sqrt{\lambda}\cdot\sin(\sqrt{\lambda}L) = 0 \iff c_1 = 0 \ \underline{\text{or}}\ \sin(\sqrt{\lambda}L) = 0. \end{aligned}$$

If we consider the case that $c_1 = 0$ then we have $F(x) = 0$ for $\lambda \in \mathbb{R}^+$ but we should try to keep as many solutions as possible and we ignore this case. Thus assume that $c_1 \neq 0$ we have that $\sin(\sqrt{\lambda}L) = 0$, which is true for $\sqrt{\lambda} = n\pi/L$ and we have the following eigenvalue/eigenfunction pairs indexed by n,

$$F_n(x) = c_n \cos(\sqrt{\lambda}x), \quad \lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, 3, \ldots. \tag{41}$$

We now consider the $\lambda \in \mathbb{R}^-$ case to find that,

$$\begin{aligned} \lambda \in \mathbb{R}^- &\ :\ F'(x) = c_3\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}0) + c_4\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}0) = c_3\cdot 0 + c_4\cdot 1 = 0 \implies c_4 = 0,\\ \lambda \in \mathbb{R}^- &\ :\ F'(x) = c_3\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}L) + c_4\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}L) = c_3\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}L) + 0\cdot\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}L) =\\ &= c_3\sqrt{|\lambda|}\frac{e^{\sqrt{|\lambda|}L} - e^{-\sqrt{|\lambda|}L}}{2} = 0 \implies c_3 = 0, \end{aligned}$$

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[5] Here the boundary conditions correspond to perfect insulation of both endpoints

[6] An insulated bar is discussed in examples 4 and 5 on page 557.

[7] This occurs in conjunction with the following argument. Since (32) must be true for all (x,t) then both sides must be equal to a function that has neither t's nor x's. Hence they must be equal to a constant function. To see that this is true put an x or t on the side that has λ and test points.

[8] These elementary methods are those you learned in ODE's and can be found in the ODE review of the lecture slides on separation of variables.

[9] We assume that $G(t) = 0$ because if it did then we would have $u(x,t) = F(x)G(t) = F(x)\cdot 0 = 0$, which is called the trivial solution and is ignored since it is already in thermal equilibrium. We care about dynamics!

which means that for $\lambda \in \mathbb{R}^-$ we only have the trivial solution $F(x) = 0$. Lastly, we consider the case $\lambda = 0$ to get,

$$\lambda = 0 \quad : \quad F'(0) = F'(L) = c_6 = 0 \implies c_5 \in \mathbb{R}, \tag{42}$$

which gives the last eigenpair,[10]

$$F_0 = c_0 \in \mathbb{R} \quad \lambda_0 = 0. \tag{43}$$

Noting that there are infinitely many λ's implies now that there are infinitely many temporal solutions (35) and we have,

$$G_n(t) = A_n e^{\lambda_n c^2 t}, \quad \lambda = \frac{n^2\pi^2}{L^2}, \ n = 1, 2, 3, \ldots. \tag{44}$$

For the case where $\lambda = 0$ we have the ODE $G'(t) = 0$, whose solution is$G_0(t) = A_0 \in \mathbb{R}$. Thus we have infinitely many functions, that solve the PDE, of the form:

$$u_n(x,t) = F_n(x)G_n(t), \ n = 0, 1, 2, 3, \ldots \tag{45}$$

Hence since the PDE is linear superposition implies that we have the general solution,

$$\begin{aligned}
u(x,t) &= \sum_{n=0}^{\infty} u_n(x,t) = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t) && (46)\\
&= F_0(x)G_0(x) + \sum_{n=1}^{\infty} F_n(x)G_n(t) && (47)\\
&= c_0 \cdot A_0 + \sum_{n=1}^{\infty} A_n c_n \cos(\sqrt{\lambda_n} x) e^{\lambda_n c^2 t} && (48)\\
&= a_0 + \sum_{n=1}^{\infty} a_n \cos(\sqrt{\lambda_n} x) e^{\lambda_n c^2 t}, && (49)
\end{aligned}$$

which is the general solution of the heat equation with the given boundary conditions.

4.2. **Qualitative Dynamics.** Describe how the long term behavior of the general solution to (28)-(31) changes as the thermal conductivity, K, is increased while all other parameters are held constant. Also, describe how the solution changes when the linear density, ρ, is increased while all other parameters are held constant.

- If κ(thermal conductivity) is increased, the temporal solution decays faster and the system reaches equilibrium sooner.
- If ρ(density) is increased, the temporal solution decays slower and the system takes longer to reach equilibrium.

4.3. **Fourier Series : Solution to the IVP.** Define,

$$f(x) = \begin{cases} \dfrac{2k}{L} x, & 0 < x \leq \frac{L}{2}, \\[2ex] \dfrac{2k}{L}(L - x), & \frac{L}{2} < x < L \end{cases} \tag{50}$$

and for the following questions we consider the solution, u, to the heat equation given by, (28)-(29), which satisfies the initial condition given by (80). [11] For $L = 1$ and $k = 1$, find the particular solution to (28)-(29) with boundary conditions (30)-(31) for when the initial temperature profile of the medium is given by (80). Show that $\lim_{t\to\infty} u(x,t) = f_{avg} = 0.5$.[12]

To find the unknown constants present in the general solution we must apply an initial condition, $u(x,0) = f(x)$. Doing so gives,

$$\begin{aligned}
u(x,0) = f(x) &= a_0 + \sum_{n=1}^{\infty} a_n \cos(\sqrt{\lambda_n} x) e^{\lambda_n c^2 \cdot 0} && (51)\\
&= a_0 + \sum_{n=1}^{\infty} a_n \cos(\sqrt{\lambda_n} x), && (52)
\end{aligned}$$

[10]Here we have used the subscripts to denote that these are all associated with the $\lambda = 0$ case. We have also trivially changed c_5 to c_0.

[11]When solving the following problems it would be a good idea to go back through your notes and the homework looking for similar calculations.

[12]It is interesting here to note that though the initial condition f doesn't appear to satisfy the boundary conditions its periodic Fourier extension does. That is, if you draw the even periodic extension of the initial condition then you would see that the slope is not well defined at the end points. Remembering that the Fourier series averages the right and left hand limits of the periodic extension of the function f at the endpoints shows that the boundary conditions are, in fact, satisfied, since the derivative of an average is the average of derivatives.

which is a Fourier cosine half-range expansion of the initial condition. Thus the unknown constants are Fourier coefficients and,

(53) $$a_0 = \frac{1}{L}\int_0^L f(x)dx,$$

(54) $$a_n = \frac{2}{L}\int_0^L f(x)\cos(\sqrt{\lambda_n}x)dx.$$

If we note that these integrals have been done in a previous homework on Fourier series, then we known these Fourier coefficients as,

(55) $$a_0 = \frac{k}{2},$$

(56) $$a_n = \frac{4k}{n^2\pi^2}\left[2\cos\left(\frac{n\pi}{2}x\right) - (-1)^n - 1\right].$$

Moreover, if we take $L = k = 1$ we see that $\lim_{t\to\infty} u(x,t) = a_0 = .5$, which implies that under these insulating boundary conditions the equilibrium state for the medium is a constant function $u = .5$ and that this is nothing more than the average of the initial configuration.

5. Wave Equation on a closed and bounded spatial domain of $\mathbb{R}^{1+1}$

Consider the one-dimensional wave equation,

(57) $$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2},$$

(58) $$x \in (0, L), \qquad t \in (0, \infty), \qquad c^2 = \frac{T}{\rho}.$$

Equations (57)-(58) model the time-evolution of the displacement,$u = u(x,t)$, from rest, of an elastic medium in one-dimension. The object, of length L, is assumed to have a homogeneous lateral tension T, and linear density ρ. That is, $T, \rho \in \mathbb{R}^+$. Assume, as well, the boundary conditions[13],

(59) $$u_x(0,t) = 0, u_x(L,t) = 0,$$

and initial conditions,

(60) $$u(x,0) = f(x),$$

(61) $$u_t(x,0) = g(x).$$

5.1. **Separation of Variables : General Solution.** Assume that the solution to (57)-(58) is such that $u(x,t) = F(x)G(t)$ and use separation of variables to find the general solution to (57)-(58), which satisfies (59)-(61). [14] [15]

The only difference between this problem and the heat problem above are the time-dynamics specified by the PDE. This gives a second-order constant linear ODE in time and from this ODE we have oscillations of Fourier modes instead of exponential decay.

Assume that, $u(x,t) = F(x)G(t)$ then $u_{xx} = F''(x)G(t)$ and $u_{tt} = F(x)G''(t)$ and the 1-D wave equation becomes,

(62) $$\frac{G''(t)}{c^2G(t)} = \frac{F''(x)}{F(x)} = -\lambda,$$

where we have introduced the separation constant λ.[16] From this equation we have the two ODE's,

(63) $$G''(t) + \lambda c^2 G(t) = 0,$$

(64) $$F''(x) + \lambda F(x) = 0.$$

[13]These boundary conditions imply that the object must have zero slope at each endpoint.

[14]It is important to notice that the solution to the spatial portion of the problem is the same as the heat problem above.

[15]Remember that in this case we have a nontrivial spatial solution for zero eigenvalue. From this you should find the associated temporal function should find that $G_0(t) = C_1 + C_2 t$.

[16]This occurs in conjunction with the following argument. Since (62) must be true for all (x,t) then both sides must be equal to a function that has neither t's nor x's. Hence they must be equal to a constant function. To see that this is true put an x or t on the side that has λ and test points.

Each of these ODE's can be solved through 'elementary methods' to get,[17]

$$\lambda \in \mathbb{R} \quad : \quad G(t) = A_1 \cos\left(c\sqrt{\lambda} t\right) + A_1^* \sin\left(c\sqrt{\lambda} t\right), \quad A_1, A_1^* \in \mathbb{R}, \tag{65}$$

$$\lambda \in \mathbb{R}^+ \quad : \quad F(x) = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x), \tag{66}$$

$$\lambda \in \mathbb{R}^- \quad : \quad F(x) = c_3 \cosh(\sqrt{|\lambda|} x) + c_4 \sinh(\sqrt{|\lambda|} x), \tag{67}$$

$$\lambda = 0 \quad : \quad F(x) = c_5 + c_6 x. \tag{68}$$

Each of the functions $F(x)$ must also satisfy the boundary conditions, $u_x(0,t) = 0$ and $u_x(L,t)$ and so we won't need all of them. Notice that the boundary conditions imply that,

$$u_x(0,t) = F'(0)G(t) = 0, \tag{69}$$

$$u_x(L,t) = F'(L)G(t) = 0, \tag{70}$$

which gives $F'(0) = 0$ and $F'(L) = 0$.[18] So, we now have to determine, which of the previous functions, $F(x)$, satisfy these boundary conditions. To this end we have the following arguments,

$$\begin{aligned}
\lambda \in \mathbb{R}^+ \quad &: \quad F'(0) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}0) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}0) = c_1\sqrt{\lambda}\cdot 0 + c_2\sqrt{\lambda}\cdot 1 \implies c_2 = 0,\\
\lambda \in \mathbb{R}^+ \quad &: \quad F'(L) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}L) = c_1\sqrt{\lambda}\cdot\sin(\sqrt{\lambda}L) + 0\cdot\sqrt{\lambda}\cos(\sqrt{\lambda}L) \implies\\
&\implies c_1\sqrt{\lambda}\cdot\sin(\sqrt{\lambda}L) = 0 \iff c_1 = 0 \ \underline{\text{or}}\ \sin(\sqrt{\lambda}L) = 0.
\end{aligned}$$

If we consider the case that $c_1 = 0$ then we have $F(x) = 0$ for $\lambda \in \mathbb{R}^+$ but we should try to keep as many solutions as possible and we ignore this case. Thus assume that $c_1 \neq 0$ we have that $\sin(\sqrt{\lambda}L) = 0$, which is true for $\sqrt{\lambda} = n\pi/L$ and we have the following eigenvalue/eigenfunction pairs indexed by n,

$$F_n(x) = c_n \cos(\sqrt{\lambda} x), \quad \lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, 3, \ldots. \tag{71}$$

We now consider the $\lambda \in \mathbb{R}^-$ case to find that,

$$\begin{aligned}
\lambda \in \mathbb{R}^- \quad &: \quad F'(x) = c_3\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}0) + c_4\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}0) = c_3\cdot 0 + c_4\cdot 1 = 0 \implies c_4 = 0,\\
\lambda \in \mathbb{R}^- \quad &: \quad F'(x) = c_3\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}L) + c_4\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}L) = c_3\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}L) + 0\cdot\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}L) =\\
&= \quad c_3\sqrt{|\lambda|}\frac{e^{\sqrt{|\lambda|}L} - e^{-\sqrt{|\lambda|}L}}{2} = 0 \implies c_3 = 0,
\end{aligned}$$

which means that for $\lambda \in \mathbb{R}^-$ we only have the trivial solution $F(x) = 0$. Lastly, we consider the case $\lambda = 0$ to get,

$$\lambda = 0 \quad : \quad F'(0) = F'(L) = c_6 = 0 \implies c_5 \in \mathbb{R}, \tag{72}$$

which gives the last eigenpair,[19]

$$F_0 = c_0 \in \mathbb{R} \quad \lambda_0 = 0. \tag{73}$$

Noting that there are infinitely many λ's implies now that there are infinitely many temporal solutions (65) and we have,

$$G_n(t) = G(t) = A_n \cos\left(c\sqrt{\lambda_n} t\right) + A_n^* \sin\left(c\sqrt{\lambda_n} t\right), \quad \lambda_n = \frac{n^2\pi^2}{L^2}, \ n = 1, 2, 3, \ldots. \tag{74}$$

For the case where $\lambda = 0$ we have the ODE $G''(t) = 0$, whose solution is$G_0(t) = A_0 + A_0^* t$. Thus we have infinitely many functions, that solve the PDE, of the form:

$$u_n(x,t) = F_n(x)G_n(t), \ n = 0, 1, 2, 3, \ldots \tag{75}$$

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[17]These elementary methods are those you learned in ODE's and can be found in the ODE review of the lecture slides on separation of variables.

[18]We assume that $G(t) = 0$ because if it did then we would have $u(x,t) = F(x)G(t) = F(x)\cdot 0 = 0$, which is called the trivial solution and is ignored since it is already in thermal equilibrium. We care about dynamics!

[19]Here we have used the subscripts to denote that these are all associated with the $\lambda = 0$ case. We have also trivially changed c_5 to c_0.

Hence since the PDE is linear superposition implies that we have the general solution,

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t) \tag{76}$$

$$= F_0(x)G_0(x) + \sum_{n=1}^{\infty} F_n(x)G_n(t) \tag{77}$$

$$= c_0 \cdot (A_0 + A_0^* t) + \sum_{n=1}^{\infty} c_n \cos(\sqrt{\lambda_n} x) \left[A_n \cos\left(c\sqrt{\lambda_n} t\right) + A_n^* \sin\left(c\sqrt{\lambda_n} t\right)\right] \tag{78}$$

$$= a_0 + a_0^* t + \sum_{n=1}^{\infty} \cos(\sqrt{\lambda_n} x) \left[a_n \cos\left(c\sqrt{\lambda_n} t\right) + a_n^* \sin\left(c\sqrt{\lambda_n} t\right)\right], \tag{79}$$

which is the general solution of the one-dimensional wave equation with the given the boundary conditions.

5.2. **Qualitative Dynamics.** Describe how the the general solution to (57)-(58) changes as the tension, T, is increased while all other parameters are held constant. Also, describe how the solution changes when the linear density, ρ, is increased while all other parameters are held constant.

- If T(Tension) is increased, the temporal solution oscillate faster.
- If ρ(density) is increased, the temporal oscillates slower.

5.3. **Fourier Series : Solution to the IVP.** Define,

$$f(x) = \begin{cases} \dfrac{2k}{L} x, & 0 < x \leq \frac{L}{2}, \\ \dfrac{2k}{L}(L-x), & \frac{L}{2} < x < L. \end{cases} \tag{80}$$

Let $L = 1$ and $k = 1$ and find the particular solution, which satisfies the initial displacement, $f(x)$, given by (80) and has zero initial velocity for all points on the object.

To find the unknown constants present in the general solution we must apply an initial condition, $u(x,0) = f(x)$. Doing so gives,

$$u(x,0) = f(x) = a_0 + a_0^* \cdot 0 + \sum_{n=1}^{\infty} \cos(\sqrt{\lambda_n} x) \left[a_n \cos\left(c\sqrt{\lambda_n} \cdot 0\right) + a_n^* \sin\left(c\sqrt{\lambda_n} \cdot 0\right)\right] \tag{81}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos(\sqrt{\lambda_n} x), \tag{82}$$

which is a Fourier cosine half-range expansion of the initial condition. Thus the unknown constants are Fourier coefficients and,

$$a_0 = \frac{1}{L} \int_0^L f(x) dx, \tag{83}$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos(\sqrt{\lambda_n} x) dx. \tag{84}$$

If we note that these integrals have been done in a previous homework on Fourier series, then we known these Fourier coefficients as,

$$a_0 = \frac{k}{2}, \tag{85}$$

$$a_n = \frac{4k}{n^2 \pi^2} \left[2 \cos\left(\frac{n\pi}{2} x\right) - (-1)^n - 1\right]. \tag{86}$$

Now we apply the initial velocity, $u_t(x,0) = g(x) = 0$ to get that,

$$u_t(x,0) = g(x) = a_0^* + \sum_{n=1}^{\infty} \cos(\sqrt{\lambda_n} x) \left[a_n c\sqrt{\lambda_n} \sin\left(c\sqrt{\lambda_n} \cdot 0\right) + a_n^* c\sqrt{\lambda_n} \cos\left(c\sqrt{\lambda_n} \cdot 0\right)\right] \tag{87}$$

$$= a_0^* + \sum_{n=1}^{\infty} a_n^* c\sqrt{\lambda_n} \cos(\sqrt{\lambda_n} x) \tag{88}$$

which is a Fourier cosine half-range expansion of the initial condition. Thus the unknown constants are Fourier coefficients and,

$$a_0^* = \frac{1}{L} \int_0^L g(x) dx, \tag{89}$$

$$a_n^* = \frac{2}{c\sqrt{\lambda_n} L} \int_0^L g(x) \cos(\sqrt{\lambda_n} x) dx, \tag{90}$$

but since $g(x) = 0$ we have that $a_0^* = a_n^* = 0$ for all n.[20]

[20]It is interesting to note that if $a_0^* \neq 0$ then the displacement grows with time. This is a consequence of the boundary conditions, which require the string to be flat at the endpoints but do not require they stay fixed.