Note Title 2/17/2010

Laplace's eqn in spherical coords

$$\nabla^2 V(r,\theta,\varphi) = 0$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\varphi^2} = 0$$

multiply thru by $r^2\sin^2\theta$

Assume $V(r,\theta,\varphi) = R(r)\,\Theta(\theta)\,\Phi(\varphi)$

$$\underbrace{\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)}_{f(r,\theta)}+\underbrace{\frac{1}{\Phi}\frac{d^2\Phi}{d^2\varphi}}_{f(\varphi)}=0$$

$$m^2 \qquad\qquad -m^2 = 0$$

separation const

$m = 0, 1, 2, \cdots$

$$\frac{d^2\Phi}{d\varphi^2} = -m^2\Phi \qquad \Phi = A e^{\pm im\varphi}$$

$$e^{\pm im\varphi} = \cos m\varphi + i\sin m\varphi$$

V is same for φ & $\varphi + 2\pi$

$$\Phi(\varphi) = A e^{\pm im\varphi} = \Phi(\varphi+2\pi) = A e^{\pm im(\varphi+2\pi)} = A e^{\pm im\varphi} e^{\pm im2\pi}$$

$e^{\pm im 2\pi} = 1$!!

For the case of azimuthal symmetry (no φ dependence)

$$\frac{d^2\Phi}{d\varphi^2} = 0 \qquad \Phi \text{ is a const} \qquad m = 0$$

$$\nabla^2 V = \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) = 0 \qquad (r^2 \text{ factored out})$$

Separation of variables

$V(r,\theta) = R(r)\,\Theta(\theta)$ & divide thru by V

$$\underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}_{k} + \underbrace{\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)}_{-k} = 0$$

$$k - k = 0$$

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - kR = 0 \qquad \boxed{R(r) = Ar^{\ell} + Br^{-(\ell+1)}}$$

$$r^2 l(l-1) r^{(l-2)} + 2r l r^{(l-1)} - k r^l = l(l-1) r^l + 2l r^l - k r^l$$

$$= 0 \quad \text{if} \quad k = l(l+1)$$

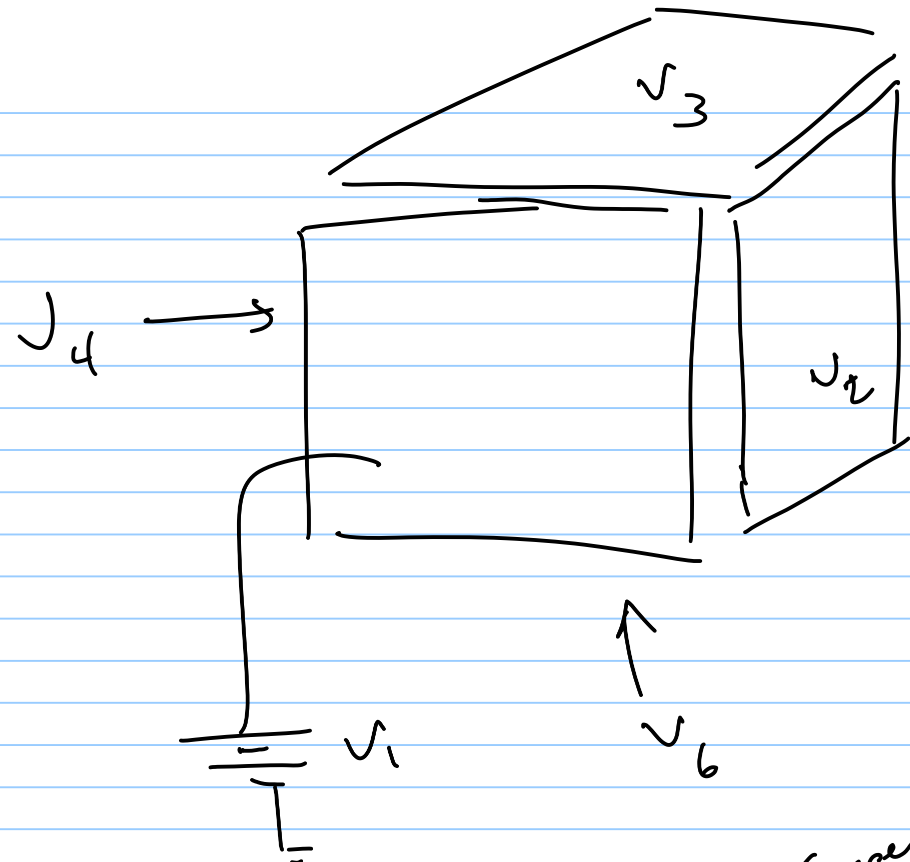
$$\Theta = P_l(\cos\theta) \quad \text{for } l \text{ integer}$$

$\uparrow$ polynomial in $\cos\theta$

$$x = \cos\theta$$
$$dx = -\sin\theta \, d\theta$$

$$P_0 = 1 \qquad P_1 = \underbrace{\cos\theta}_{x} \qquad P_2 = \frac{1}{2}(3\underbrace{\cos^2\theta}_{x^2} - 1)$$

$$\int_{-1}^{1} P_l(x) P_m(x) \, dx = \underbrace{\int_{\pi}^{0} P_l(\cos\theta) P_m(\cos\theta)(-\sin\theta \, d\theta)}_{\begin{array}{ll} 0 & \text{if } l \neq m \\ \frac{2}{2m+1} & \text{if } l = m \end{array}}$$



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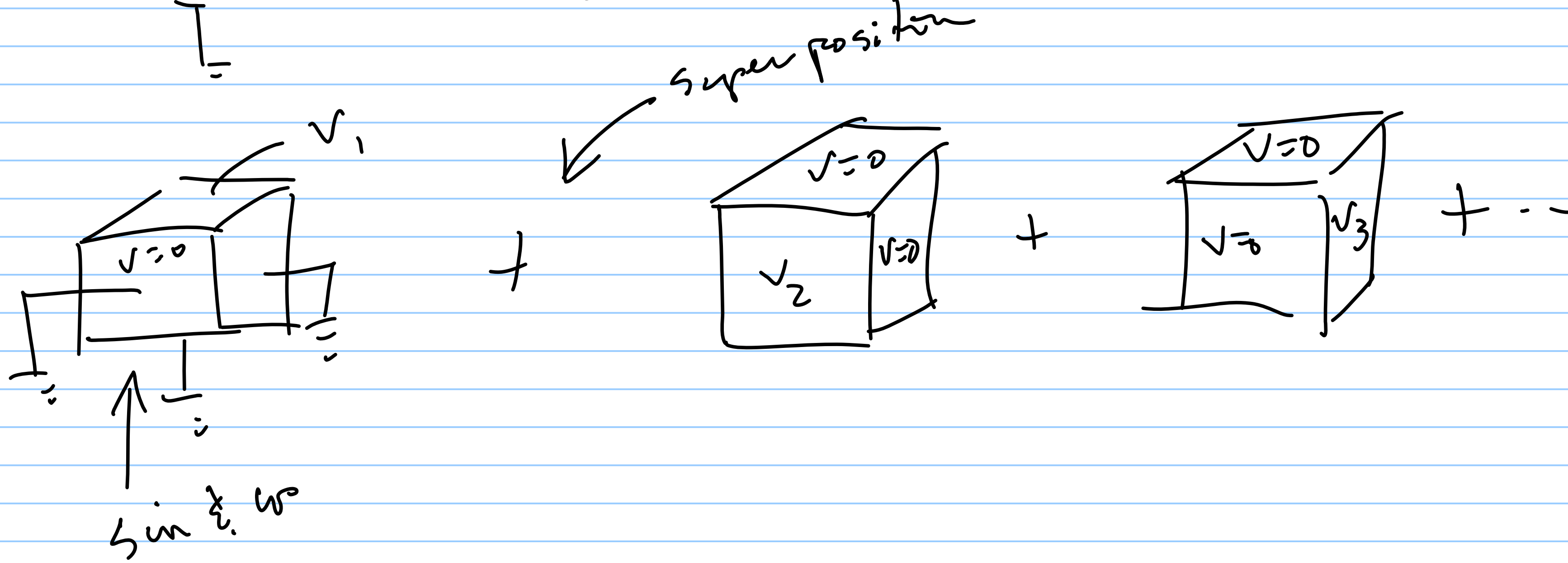

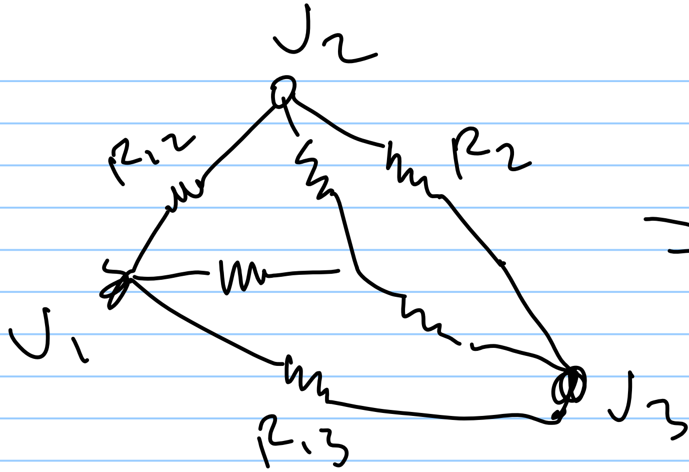
J2
R12
R2
V1
J3
R13

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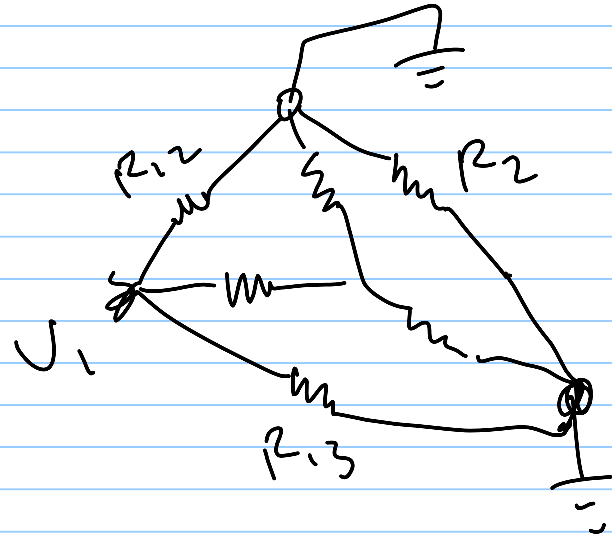
R12
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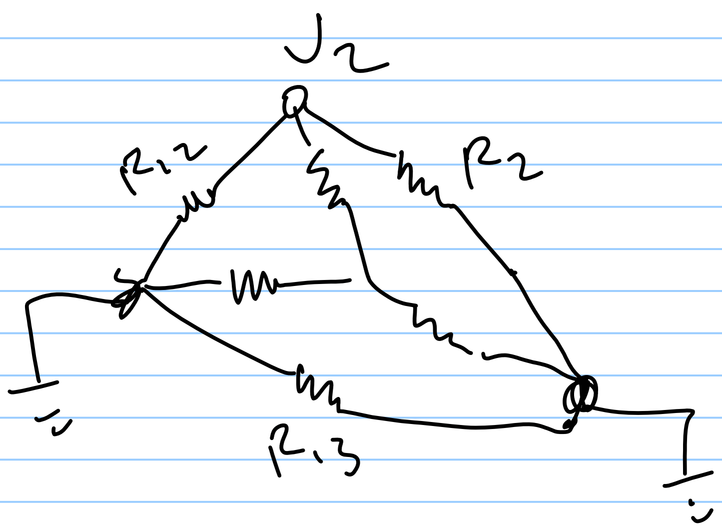
J2
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