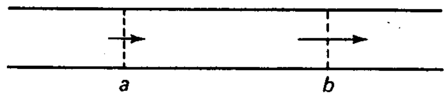

Figure 60-1 Cars entering and leaving a segment of roadway.

If there are no entrances nor exits on this road, then the number of cars between $x = a$ and $x = b$ might still change in time. The number decreases due to cars leaving the region at $x = b$, and the number increases as a result of cars entering the region at $x = a$. Assuming that no cars are created or destroyed in between, then the changes in the number of cars result from crossings at $x = a$ and $x = b$ only. If cars are flowing at the rate of 300 cars per hour at $x = a$, but flowing at the rate of 275 cars per hour at $x = b$, then clearly the number of cars between $x = a$ and $x = b$ is increasing by 25 cars per hour. We can generalize this result to situations in which the number of cars crossing each boundary (the traffic flow $q(a, t)$ and $q(b, t)$) is not constant in time. The rate of change of the number of cars, dN/dt, equals the number per unit time crossing at $x = a$ (moving to the right) minus the number of cars per unit time crossing (again moving to the right) at $x = b$, or

$$\boxed{\frac{dN}{dt} = q(a, t) - q(b, t),} \tag{60.2}$$

since the number of cars per unit time is the flow $q(x, t)$.

Perhaps this derivation of this important result was not clear to some of you. An alternate derivation of this result follows. The difference in number of cars between times $t + \Delta t$ and t, $N(t + \Delta t) - N(t)$, equals the number crossing at $x = a$ between $t + \Delta t$ and t, which for Δt small is approximately $q(a, t)\,\Delta t$, minus the number crossing at $x = b$ between $t + \Delta t$ and t, which for Δt small is approximately $q(b, t)\Delta t$. Thus,

$$N(t + \Delta t) - N(t) \approx \Delta t(q(a, t) - q(b, t)).$$

Dividing by Δt and taking the limit as $\Delta t \to 0$ again yields equation 60.2. We improve this last derivation by eliminating the need for using an approximation. Consider the difference between the number of cars in the region at $t = t_0$ and $t = t_1$ (these times do not need to be near each other). An exact expression is needed for the number of cars crossing at $x = b$ between $t = t_0$ and $t = t_1$. Since $q(b, t)$ is the number crossing at $x = b$ per unit time, then $\int_{t_0}^{t_1} q(b, t)\, dt$ is the number crossing at $x = b$ between $t = t_0$ and $t = t_1$. In the approximate derivation, $t = t_1$ was near $t = t_0$ and this integral was

approximated by $\Delta t\, q(b, t_0)$. However, without an approximation

$$N(t_1) - N(t_0) = \int_{t_0}^{t_1} q(a, t)\, dt - \int_{t_0}^{t_1} q(b, t)\, dt = \int_{t_0}^{t_1} (q(a, t) - q(b, t))\, dt.$$

Divide this expression by $t_1 - t_0$ and take the limit as t_1 tends to t_0. Equivalently, (but slightly more elegantly) take the derivative with respect to t_1. Since t_0 does not depend on t_1 (they are two independent times), we obtain

$$\frac{dN(t_1)}{dt_1} = \frac{d}{dt_1} \int_{t_0}^{t_1} (q(a, t) - q(b, t))\, dt.$$

From the Fundamental Theorem of Calculus (the theorem that implies that the derivative of the integral of $f(x)$ is $f(x)$ itself), it follows that

$$\frac{dN(t_1)}{dt_1} = q(a, t_1) - q(b, t_1).$$

Since t_1 could be any arbitrary time, t_1 is replaced in notation by t, and thus the previously stated result equation 60.2 is rederived.

Combining equations 60.1 and 60.2, yields

$$\boxed{\frac{d}{dt} \int_a^b \rho(x, t)\, dx = q(a, t) - q(b, t).} \tag{60.3}$$

This equation expresses the fact that changes in the number of cars are due only to the flow across the boundary. No cars are created or destroyed; the number of cars is conserved. This does not mean the number of cars between $x = a$ and $x = b$ is constant (if that were true then $(d/dt) \int_a^b \rho(x, t)\, dx = 0$ or $q(a, t) = q(b, t)$). Equation 60.3 is called a *conservation law in integral form* or, more concisely, an **integral conservation law**. This law expresses a property of traffic over a finite length of roadway $a \leq x \leq b$.

As an example, consider an extremely long highway which we model by a highway of infinite length. Let us assume that the flow of cars approaches zero as x approaches both $\pm\infty$,

$$\lim_{x \to \pm\infty} q(x, t) = 0.$$

From equation 60.3, it follows that

$$\frac{d}{dt} \int_{-\infty}^{\infty} \rho(x, t)\, dx = 0.$$

Integrating this yields

$$\int_{-\infty}^{\infty} \rho(x, t)\, dx = \text{constant},$$