

MATH 348 - SPRING 2008

HOMEWORK 5

1) CONSIDER THE ODE.

$$y'' + 9y = f(t) \quad (1)$$

$$f(t) = |t| \quad -\pi \leq t < \pi, \quad f(t+2\pi) = f(t) \quad (2)$$

(a) DETERMINE THE REAL FOURIER SERIES REPRESENTATION OF $f(t)$.

$|t|$ IS THE EVEN EXTENSION OF $f(t) = t$, THEREFORE THE FOURIER COSINE SERIES IS

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nt)$$

$$a_0 = \frac{1}{\pi} \int_0^{\pi} t dt = \frac{1}{2\pi} t^2 \Big|_0^{\pi} = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} t \cos(nt) dt = \frac{2}{\pi} \left[\frac{1}{n} \sin(nt) + \frac{1}{n^2} \cos(nt) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{(-1)^{n-1}}{n^2} \right] = \begin{cases} 0 & \text{FOR } n \text{ IS EVEN} \\ \frac{-4}{\pi n^2} & \text{FOR } n \text{ IS ODD} \end{cases}$$

$$= \frac{-4}{\pi (2n-1)^2}$$

$$f(t) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{4}{\pi (2n-1)^2} \cos(2n-1)t$$

(b) IF YOU WERE TO USE THE METHOD OF UNDETERMINED COEFFICIENTS, WHAT WOULD YOUR CHOICE BE FOR $y_p(t)$?

$$y_p(t) = A + \sum_{\substack{n=1 \\ n \neq 3}}^{\infty} [B_n \cos(2n-1)t + C_n \sin(2n-1)t] + B_3 t \cos(2n-1)t + C_3 t \sin(2n-1)t$$

(c) WHAT IS THE PARTICULAR SOLUTION ASSOCIATED WITH THE THIRD FOURIER MODE OF THE FORCING FUNCTION?

$$y_p(t) = A + B_3 t \cos(3t) + C_3 t \sin(3t)$$

(d) WHAT IS THE LONG TERM BEHAVIOR OF THE SOLUTION? WHAT IF THE O.D.E. HAD THE FORM $y'' + 4y = f(t)$?

THE SOLUTION DIVERGES. IF THE O.D.E. WAS $y'' + 4y = f(t)$, THE SOLUTION WOULD NOT DIVERGE BECAUSE $2n-1$ WILL NEVER EQUAL 2 AND THE PARTICULAR SOLUTION WILL NEVER EQUAL THE HOMOGENEOUS SOLUTION.

MATH 348 - SPRING 2008

HOMEWORK 5

2 GIVEN: $my'' + cy' + ky = f(t)$

(a) SHOW THAT THE FREQUENCY RESPONSE FUNCTION IS GIVEN

$$\text{BY } \hat{g}(\omega) = \frac{1}{k + i\omega c - m\omega^2} \left(\frac{1}{\sqrt{2\pi}} \right)$$

$$mg'' + cg' + kg = \delta(t)$$

$$\mathcal{F}\{mg'' + cg' + kg\} = \mathcal{F}\{\delta(t)\}$$

$$- \omega^2 m \hat{g}(\omega) + i\omega c \hat{g}(\omega) + k \hat{g}(\omega) = \frac{1}{\sqrt{2\pi}}$$

$$\hat{g}(\omega) (-\omega^2 m + i\omega c + k) = \frac{1}{\sqrt{2\pi}}$$

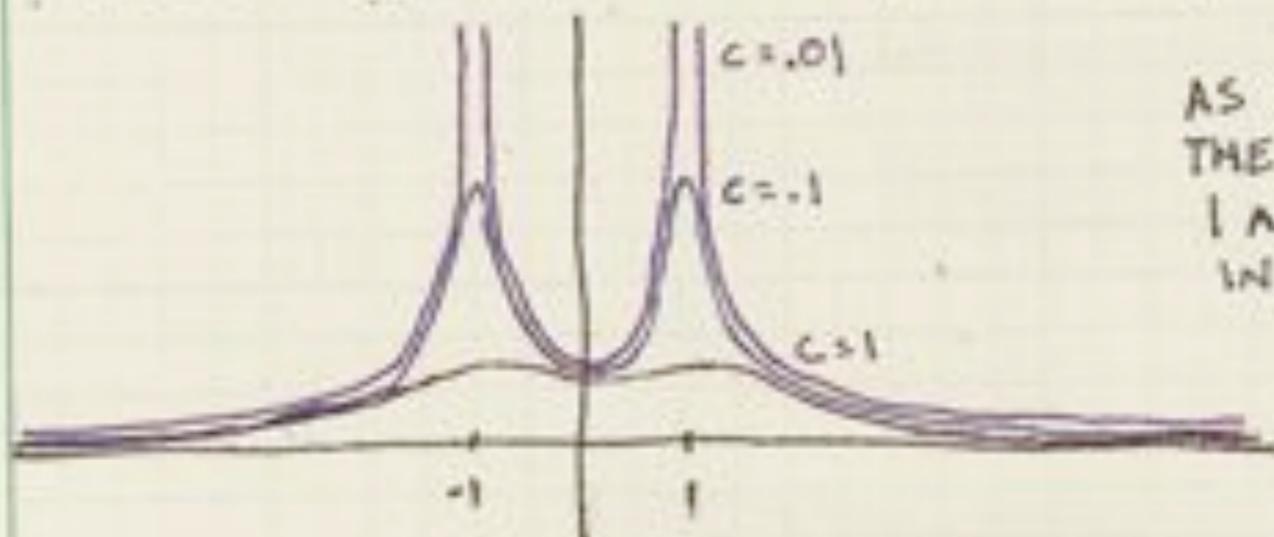
$$\hat{g}(\omega) = \frac{1}{\sqrt{2\pi} (k + i\omega c - \omega^2 m)}$$

(b) SHOW THAT FOR $m=k=1$ $|\hat{g}(\omega)| = \sqrt{\hat{g}(\omega) \overline{\hat{g}(\omega)}} = \frac{1}{\sqrt{(\omega^2-1)^2 + c^2\omega^2}} \left(\frac{1}{\sqrt{2\pi}} \right)$

$$|\hat{g}(\omega)| = \sqrt{\frac{1}{(\sqrt{2\pi})^2 ((1-\omega^2) + i\omega c) ((1-\omega^2) - i\omega c)}}$$

$$= \frac{1}{\sqrt{2\pi} \sqrt{(\omega^2-1)^2 + c^2\omega^2}}$$

(c) PLOT $|\hat{g}(\omega)|$ FOR $c^2 = \{.01, .1, 1\}$



AS c^2 APPROACHES 0,
THE SPIKES AT
1 AND -1 APPROACH
INFINITY.

MATH 349 - SPRING 2008

HOMEWORK 5

- 3] ASSUME THAT YOU ARE GIVEN FOUR SAMPLE VALUES REPRESENTED BY $f = [0, 1, 4, 9]^T$. CALCULATE THE FOURIER MATRIX, F_4 AND THE DISCRETE FOURIER TRANSFORM, \hat{f} .

$$F_4 = \begin{bmatrix} w^0 & w^0 & w^0 & w^0 \\ w^0 & w^1 & w^2 & w^3 \\ w^0 & w^2 & w^4 & w^6 \\ w^0 & w^3 & w^6 & w^9 \end{bmatrix}$$

$$w = e^{-\frac{2\pi i}{N}} = e^{-\frac{\pi}{2}i} = -i$$

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$$

$$\hat{f} = F_4 f = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 14 \\ -4 + 8i \\ -6 \\ -4 - 8i \end{bmatrix}$$

- 4] COMPUTE THE FOURIER TRANSFORM OF THE PREVIOUS SAMPLE USING THE FAST FOURIER TRANSFORM

MATH 348 - SPRING 2008

HOMEWORK 5

- 5 (a) EXPLAIN THE PURPOSE OF THE MODIFIERS DISCRETE, FAST AND FRACTIONAL.

THE DISCRETE MODIFIER IS USED BECAUSE THE DFT REQUIRES A DISCRETE INPUT.

THE FAST MODIFIER IS USED BECAUSE THE FFT IS AN EFFICIENT ALGORITHM TO CALCULATE A DFT.

THE FRACTIONAL MODIFIER IS USED BECAUSE THE FRFT CAN TRANSFORM A FUNCTION TO AN INTERMEDIATE (OR FRACTIONAL) DOMAIN BETWEEN TIME AND FREQUENCY.

- (b) IF AUDIO SIGNALS ARE CONTINUOUS, THEN WHY SHOULD ONE STUDY DISCRETE FOURIER TRANSFORMS?

THE DFT IS IDEAL FOR SAMPLED SIGNALS. SAMPLING A SIGNAL REDUCES THE AMOUNT OF DATA THAT NEEDS TO BE STORED.

- (c) HOW ARE DIGITAL SPECTRUM ANALYZERS RELATED TO THE DFT? WHAT IS THE DATA OUTPUT OF A DIGITAL SPECTRUM ANALYZER?

A DIGITAL SPECTRUM ANALYZER COMPUTES THE DFT TO TRANSFORM A WAVE FORM INTO THE COMPONENTS OF ITS FREQUENCY SPECTRUM

A SPECTRUM ANALYZER DISPLAYS A POWER SPECTRUM OVER A GIVEN FREQUENCY RANGE

- (d) EXPLAIN THE DIFFERENCE BETWEEN THE FRACTION AND NON-FRACTION FOURIER TRANSFORMS IN TERMS OF THE RECTANGULAR AND SINC FUNCTIONS.

WITH THE FOURIER TRANSFORM, A SQUARE WAVE IN THE TIME DOMAIN BECOMES THE SINC FUNCTION IN THE FREQUENCY DOMAIN. A FRACTIONAL FOURIER TRANSFORM CAN TRANSFORM A SQUARE WAVE INTO A FUNCTION THAT IS BETWEEN A SQUARE WAVE AND A SINC FUNCTION.