Solutions of scalar wave equation

- 2nd order PDE: $$\frac{\partial^2}{\partial z^2}\psi(z,t)-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(z,t)=0$$
  - Assume separable solution $\psi(z,t)=f\left(z\right)g\left(t\right)$
  - 2 solutions for f(z), g(t)
  - Full solution is a linear combination of both solutions

$$\psi(z,t)=f\left(z\right)g\left(t\right)=\left(A_1\cos kz+A_2\sin kz\right)\left(B_1\cos\omega t+B_2\sin\omega t\right)$$

  - Equivalent representation:

$$\psi(z,t)=A_1\cos\left(kz+\omega t+\phi_1\right)+A_2\cos\left(kz-\omega t+\phi_2\right)$$

forward propagating + backward propagating waves

- Complex (phasor) representation:

$$\psi(z,t)=\mathrm{Re}\left[a\,e^{i\left(kz-\omega t+\phi\right)}\right] \quad \text{or} \quad \psi(z,t)=\mathrm{Re}\left[A\,e^{i\left(kz-\omega t\right)}\right]$$

Here *A* is complex, includes phase

Maxwell's Equations to wave eqn

- The induced polarization, **P**, contains the effect of the medium:

$$\vec{\nabla}\cdot\mathbf{E}=0 \qquad \vec{\nabla}\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$$

$$\vec{\nabla}\cdot\mathbf{B}=0 \qquad \vec{\nabla}\times\mathbf{B}=\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}+\mu_0\frac{\partial\mathbf{P}}{\partial t}$$

Take the curl:

$$\vec{\nabla}\times\left(\vec{\nabla}\times\mathbf{E}\right)=-\frac{\partial}{\partial t}\vec{\nabla}\times\mathbf{B}=-\frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}+\mu_0\frac{\partial\mathbf{P}}{\partial t}\right)$$

Use the vector ID:

$$\mathbf{A}\times\left(\mathbf{B}\times\mathbf{C}\right)=\mathbf{B}\left(\mathbf{A}\cdot\mathbf{C}\right)-\mathbf{C}\left(\mathbf{A}\cdot\mathbf{B}\right)$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\mathbf{E}\right)=\vec{\nabla}\left(\vec{\nabla}\cdot\mathbf{E}\right)-\left(\vec{\nabla}\cdot\vec{\nabla}\right)\mathbf{E}=-\vec{\nabla}^2\mathbf{E}$$

$$\boxed{\vec{\nabla}^2\mathbf{E}-\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}=\mu_0\frac{\partial^2\mathbf{P}}{\partial t^2}}$$

"Inhomogeneous Wave Equation"

Maxwell's Equations in a Medium

- The induced polarization, **P**, contains the effect of the medium:

$$\vec{\nabla}^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2}$$

- Sinusoidal waves of all frequencies are solutions to the wave equation
- The polarization (**P**) can be thought of as the driving term for the solution to this equation, so the polarization determines which frequencies will occur.
- For linear response, **P** will oscillate at the same frequency as the input.

$$\mathbf{P}(\mathbf{E}) = \varepsilon_0 \chi \mathbf{E}$$

- In nonlinear optics, the induced polarization is more complicated:

$$\mathbf{P}(\mathbf{E}) = \varepsilon_0 \left(\chi^{(1)}\mathbf{E} + \chi^{(2)}\mathbf{E}^2 + \chi^{(3)}\mathbf{E}^3 + \ldots \right)$$

- The extra nonlinear terms can lead to new frequencies.

Solving the wave equation:
linear induced polarization

For low irradiances, the polarization is proportional to the incident field:

$$\mathbf{P}(\mathbf{E}) = \varepsilon_0 \chi \mathbf{E}, \quad \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 (1+\chi)\mathbf{E} = \varepsilon \mathbf{E} = n^2 \mathbf{E}$$

In this simple (and most common) case, the wave equation becomes:

$$\vec{\nabla}^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{1}{c^2}\chi \frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad \rightarrow \vec{\nabla}^2 \mathbf{E} - \frac{n^2}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

Using: $\varepsilon_0 \mu_0 = 1/c^2$ $\qquad$ $\varepsilon_0(1+\chi) = \varepsilon = n^2$

The electric field is a vector function in 3D, so this is actually 3 equations:

$$\vec{\nabla}^2 E_x(\mathbf{r},t) - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2} E_x(\mathbf{r},t) = 0$$

$$\vec{\nabla}^2 E_y(\mathbf{r},t) - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2} E_y(\mathbf{r},t) = 0$$

$$\vec{\nabla}^2 E_z(\mathbf{r},t) - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2} E_z(\mathbf{r},t) = 0$$

Plane wave solutions for the wave equation

If we assume the solution has no dependence on x or y:

$$\vec{\nabla}^2\mathbf{E}(z,t)=\frac{\partial^2}{\partial x^2}\mathbf{E}(z,t)+\frac{\partial^2}{\partial y^2}\mathbf{E}(z,t)+\frac{\partial^2}{\partial z^2}\mathbf{E}(z,t)=\frac{\partial^2}{\partial z^2}\mathbf{E}(z,t)$$

$$\rightarrow \frac{\partial^2\mathbf{E}}{\partial z^2}-\frac{n^2}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} \quad = \quad 0$$

The solutions are oscillating functions, for example

$$\mathbf{E}(z,t)=\hat{\mathbf{x}}\,E_x\cos\left(k_z z-\omega t\right)$$

Where $\quad \omega = k\,c, \quad k = 2\pi n/\lambda, \quad v_{ph} = c/n$

This is a *linearly* polarized wave.
For a plane wave **E** is perpendicular to **k**, so **E** can also point in y-direction

Complex notation for EM waves

- Write cosine in terms of exponential

$$\mathbf{E}(z,t) = \hat{\mathbf{x}} E_x \cos(kz - \omega t + \phi) = \hat{\mathbf{x}} E_x \frac{1}{2}\left(e^{i(kz-\omega t+\phi)} + e^{-i(kz-\omega t+\phi)}\right)$$

  – Note E-field is a *real* quantity.
    - It is convenient to work with just one component

  – Method 1:

$$\mathbf{E}(z,t) = \hat{\mathbf{x}} \,\mathrm{Re}\left[A\, e^{i(kz-\omega t)}\right] \qquad A = E_x e^{i\phi}$$

  – Method 2:

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}\left(A\, e^{i(kz-\omega t)} + c.c.\right) \qquad A = \tfrac{1}{2} E_x e^{i\phi}$$

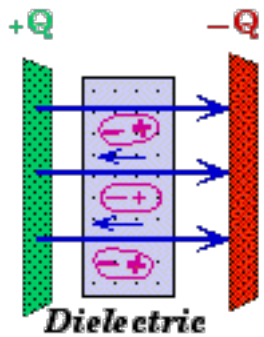
    - In *nonlinear* optics, we have to explicitly include conjugate term. Leads to extra factor of ½.

Wave energy and intensity

- Both E and H fields have a corresponding energy density (J/m^3)
  - For static fields (e.g. in capacitors) the energy density can be calculated through the work done to set up the field

$$\rho = \tfrac{1}{2}\varepsilon E^2 + \tfrac{1}{2}\mu H^2$$

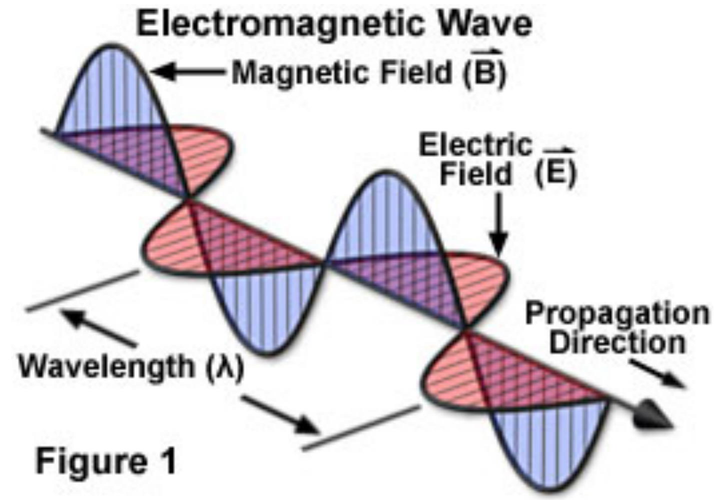
  - Some work is required to polarize the medium
  - Energy is contained in both fields, but H field can be calculated from E field



H field from E field

- H field for a propagating wave is *in phase* with E-field

$$\mathbf{H} = \hat{\mathbf{y}} H_0 \cos\left(k_z z - \omega t\right)$$
$$= \hat{\mathbf{y}} \frac{k_z}{\omega \mu_0} E_0 \cos\left(k_z z - \omega t\right)$$



- Amplitudes are not independent

$$H_0 = \frac{k_z}{\omega \mu_0} E_0 \qquad k_z = n\frac{\omega}{c} \qquad c^2 = \frac{1}{\mu_0 \varepsilon_0} \rightarrow \frac{1}{\mu_0 c} = \varepsilon_0 c$$

$$H_0 = \frac{n}{c\mu_0} E_0 = n\varepsilon_0 c E_0$$

Energy density in an EM wave

- Back to energy density, non-magnetic

$$\rho = \tfrac{1}{2}\varepsilon E^2 + \tfrac{1}{2}\mu_0 H^2$$

$$H = n\varepsilon_0 cE$$

$$\varepsilon = \varepsilon_0 n^2$$

$$\rho = \tfrac{1}{2}\varepsilon_0 n^2 E^2 + \tfrac{1}{2}\mu_0 n^2 \varepsilon_0{}^2 c^2 E^2$$

$$\mu_0 \varepsilon_0 c^2 = 1$$

$$\rho = \varepsilon_0 n^2 E^2 = \varepsilon_0 n^2 E^2 \cos^2\left(k_z z - \omega t\right)$$

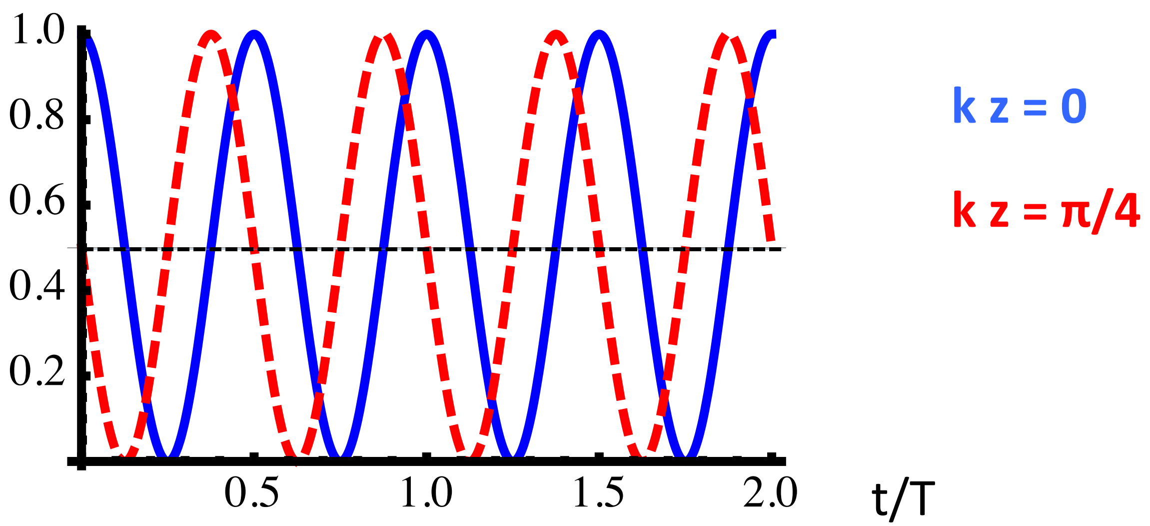
Equal energy in both components of wave

Cycle-averaged energy density

- Optical oscillations are faster than detectors
- Average over one cycle:

$$\langle \rho \rangle = \varepsilon_0 n^2 E_0{}^2 \frac{1}{T} \int_0^T \cos^2\left(k_z z - \omega t\right) dt$$

  – Graphically, we can see this should = ½



  – Regardless of position z

$$\langle \rho \rangle = \frac{1}{2} \varepsilon_0 n^2 E_0{}^2$$

Intensity and the Poynting vector

- Intensity is an energy flux (J/s/cm^2)
- In EM the Poynting vector give energy flux

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

  - For our plane wave,

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = E_0 \cos(k_z z - \omega t)\, n\varepsilon_0 c E_0 \cos(k_z z - \omega t)\, \hat{\mathbf{x}} \times \hat{\mathbf{y}}$$

$$\mathbf{S} = n\varepsilon_0 c E_0^2 \cos^2(k_z z - \omega t)\, \hat{\mathbf{z}}$$

  - **S** is along **k**

- Time average: $\mathbf{S} = \tfrac{1}{2} n \varepsilon_0 c E_0^2 \hat{\mathbf{z}}$
- *Intensity* is the magnitude of **S**

$$I = \frac{1}{2} n\varepsilon_0 c E_0^2 = \frac{c}{n}\rho = V_{phase} \cdot \rho$$

Photon flux: $F = \dfrac{I}{h\nu}$

Calculating intensity with complex wave representation

- Using the convention that we work with the complex form, with the field being the real part

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}\,\mathrm{Re}\left[A\,e^{i(kz-\omega t)}\right] \qquad A = E_x e^{i\phi}$$

  - Or write

$$\mathbf{E}(z,t) = \mathbf{E_0} e^{i(kz-\omega t)}$$

$\mathbf{E_0}$ complex, vector

  - take the real part when we want the *field*

- Time-averaged intensity

$$I = \frac{1}{2} n \varepsilon_0 c\, \mathbf{E_0} \cdot \mathbf{E_0}^*$$

  - Notice this is the sum of intensities for the different polarization components

Parseval's theorem

FT gives a different *representation* of the signal.
Energy must be conserved.

$$\int |f(t)|^2 dt = \frac{1}{2\pi}\int |F(\omega)|^2 d\omega$$

$$= \frac{1}{2\pi}\int \left[\int f(t) e^{i\omega t}\, dt\right]\left[\int f(t') e^{i\omega t'}\, dt'\right]^* d\omega$$

Note *independent* integrals for t, t'
Apply conjugation inside integral

$$= \frac{1}{2\pi}\int \left[\int f(t) e^{i\omega t}\, dt\right]\left[\int f^*(t') e^{-i\omega t'}\, dt'\right] d\omega$$

Gather ω terms into one integral.

$$= \int dt\, f(t) \int dt'\, f^*(t') \left(\frac{1}{2\pi}\int e^{i\omega(t-t')}\, d\omega\right) == \int dt\, f(t) \int dt'\, f^*(t')\, \delta(t'-t)$$

$$= \int dt\, f(t)\, f^*(t)$$

Convolution theorem

FT of the product of two functions is the convolution of the transforms

$$FT\{f(t)g(t)\} = \frac{1}{2\pi} F(\omega) \otimes G(\omega)$$

$$FT\{f(t)g(t)\} = \int f(t) g(t) e^{i\omega t}\, dt$$

$$= \int f(t) \left[\frac{1}{2\pi} \int G(\omega') e^{-i\omega' t}\, d\omega' \right] e^{i\omega t}\, dt$$

Note *independent* variables for ω, ω'

$$= \frac{1}{2\pi} \int G(\omega') d\omega' \int f(t) e^{i(\omega-\omega')t}\, dt$$

Swap order of integration: t first

$$= \frac{1}{2\pi} \int F(\omega - \omega') G(\omega') d\omega' = \frac{1}{2\pi} F(\omega) \otimes G(\omega)$$

Inverse FT of the product of two functions is the convolution of the transforms

$$FT^{-1}\{F(\omega)G(\omega)\} = f(t) \otimes g(t)$$

Graphical approach to convolution

Input functions

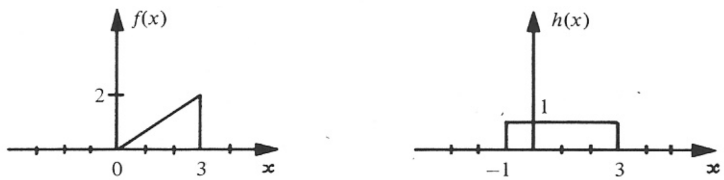


Figure 6-1 Functions used to illustrate convolution by graphical methods.

output

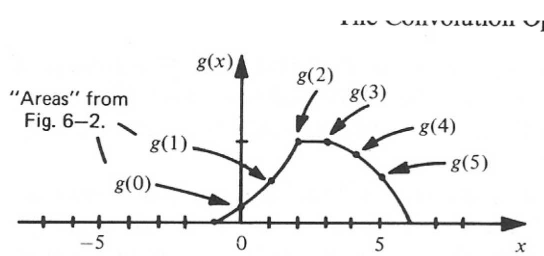


Figure 6-3 Resulting convolution of functions shown in Fig. 6-1.

Graphical view

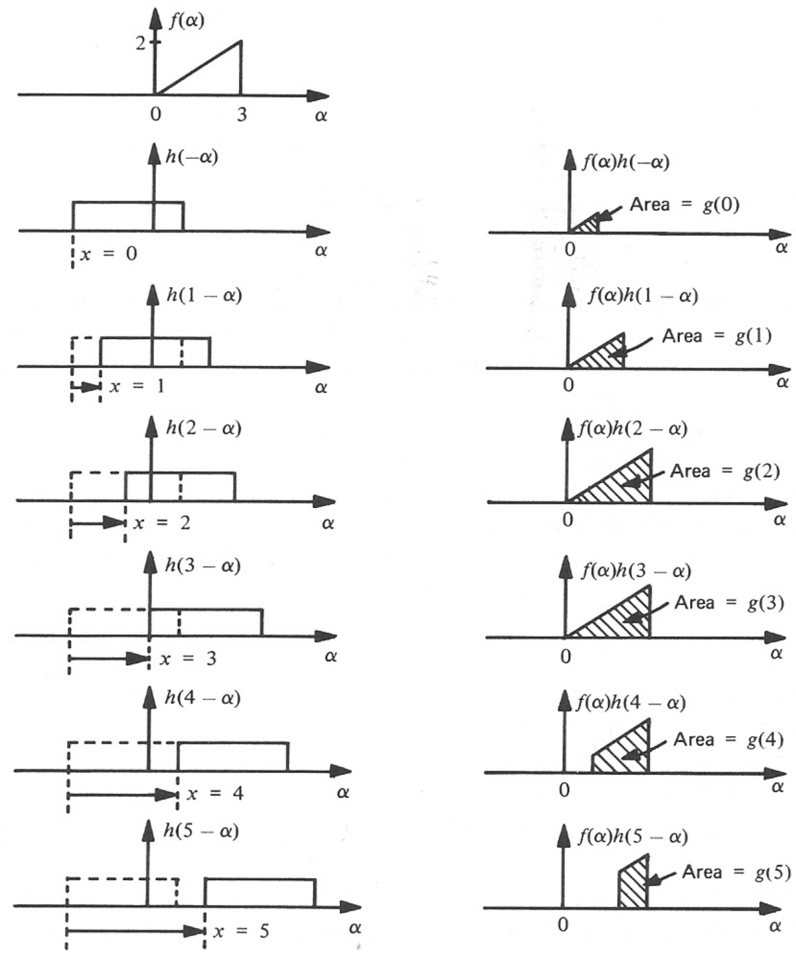


Figure 6-2 Graphical method for convolving functions of Fig. 6-1.

Smoothing effects

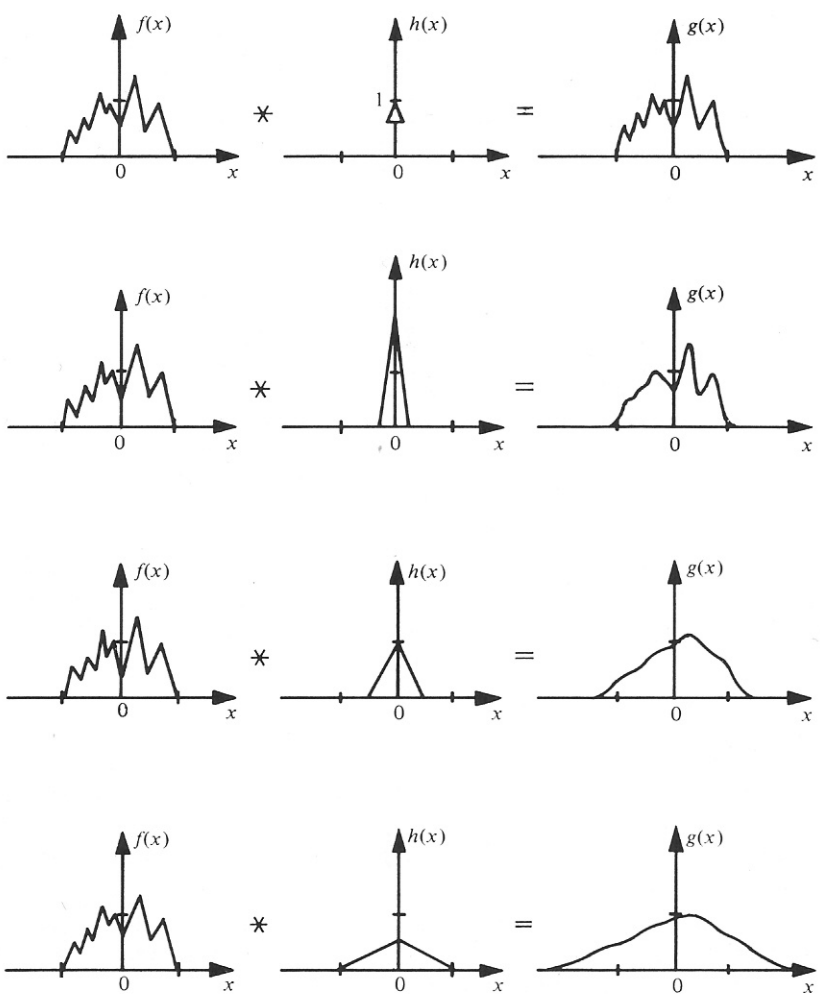


Figure 6-5 Smoothing effects of convolution.

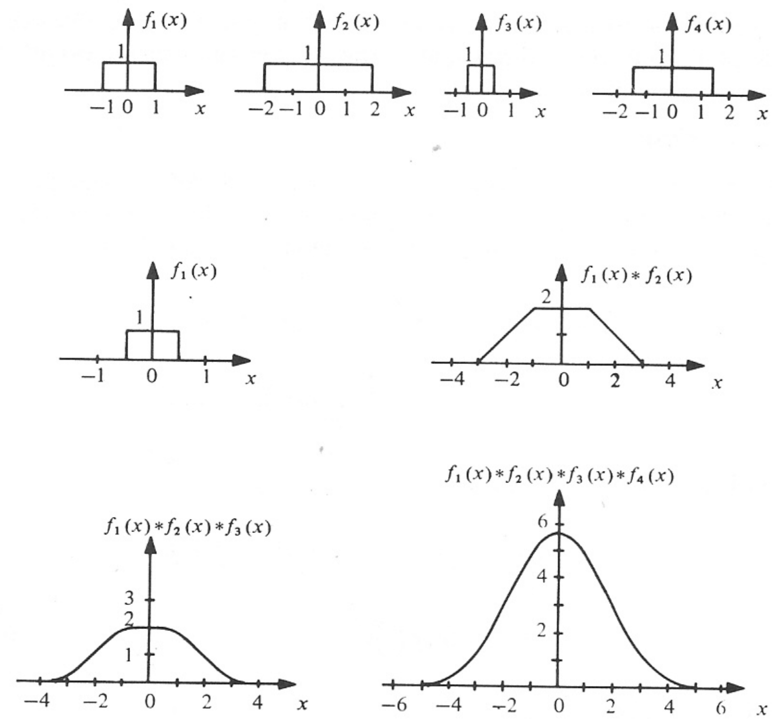


Figure 6-7 Repeated convolution of four rectangle functions.

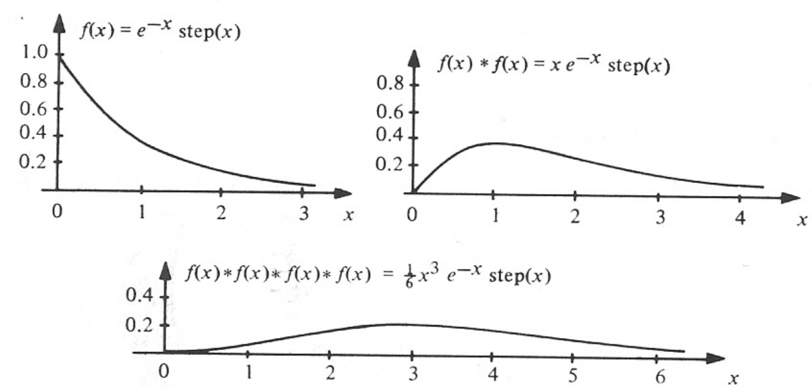


Figure 6-8 Repeated convolution of the function $\exp\{-x\}$step(x).