MATH 348 - Advanced Engineering Mathematics January 19, 2009

Homework 2, Spring 2009 **Due**: January 26, 2009

Matrix Algebra - Row Reduction - Solutions to Linear Systems

1. Find the interpolating polynomial $p(t) = a_0 + a_1 t + a_2 t^2$ for the data $(1, 12)$, $(2, 15)$, $(3, 16)$.[1] Noting that the system is linear in the coefficient data, we seek to find a_0, a_1 and a_2 that satisfies,

$$a_0 + a_1(1) + a_2(1)^2 = 12 \tag{1}$$

$$a_0 + a_1(2) + a_2(2)^2 = 15 \tag{2}$$

$$a_0 + a_1(3) + a_2(3)^2 = 16 \tag{3}$$

2. It is common to think about the equation $\mathbf{Ax} = \mathbf{b}$ as a transformation of the vector $\mathbf{x}$ to a new vector $\mathbf{b}$ given by the matrix multiplication $\mathbf{Ax}$. In this way every matrix can be thought of as a linear transformation applied to vectors.[2] Probably the most common vector transformation is that of a rotation, which in $\mathbb{R}^2$ is given by:

$$\mathbf{A} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \tag{4}$$

Let $\mathbf{x} = [1\ 0]^{\mathrm{T}}$. Describe or draw the results of the linear transformation $\mathbf{Ax}$ for $\theta \in \left\{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi\right\}$. How would these results change if $\mathbf{A} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$?

3. Given,

$$\mathbf{A} = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}.$$

Determine $\mathbf{A}^{-1}$ via:

(a) Calculate $\det(\mathbf{A})$.

(b) The Gauss-Jordan Method (pg.317).

(c) The cofactor representation (Theorem 2 pg.318).

(d) Check your result by showing $\mathbf{AA}^{-1} = \mathbf{I}$

4. Given the following for matrices:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} a + kc & b + kd \\ c & d \end{bmatrix}.$$

Calculate the determinants of the previous matrices. In each case, state the row operation used on $\mathbf{A}$ and describe how the row operation effects the determinant.

5. The determinant has a geometric interpretation. In $\mathbb{R}^2$, $\det(\mathbf{A})$ is the area of the parallelogram formed by the two vectors $\mathbf{a}_1, \mathbf{a}_2$, where $\mathbf{A} = [\mathbf{a}_1\, \mathbf{a}_2]$. In $\mathbb{R}^3$, $\det(\mathbf{A})$ is the volume of the parallelepiped formed by the three vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, where $\mathbf{A} = [\mathbf{a}_1\, \mathbf{a}_2\, \mathbf{a}_3]$. For an illustration of these objects please see the PDF's posted on blackboard.

Using the concept of volume, explain why the determinant of a 3×3 matrix $\mathbf{A}$ is zero if and only if $\mathbf{A}$ is not invertable.[3]

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[1] An interpolating polynomial for a data set is a polynomial whose graph passes through every point in the data set.

[2] See `http://en.wikipedia.org/wiki/Transformation_matrix` for more information.

[3] If the three vectors, $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, form a parrallelepiped with zero volume then what can be said about their geometric configuration? If a matrix is not invertible then what can be said about the linear independence of the rows or columns?