Abstract Vector Spaces

Introduction to Infinite Dimensional Spaces

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Overview/Keywords/References

Advanced Engineering Mathematics **Slide Set Four**

Abstract Vectors Spaces, $L_2\left(a,b\right)$, Periodic Extension, Fourier Basis

Examples:N/A

- See Also:
  - EK : 7.9 page 324-326, 11.1-11.3
- Begin:
  - 09.LN.Introduction to Fourier Series : Review of Symmetric and Periodic Functions
  - Homework 4

Before We Begin

Quote of Slide Set Four
Shannon McFarland: Give me a rampant intellectualism as a coping mechanism.
Chuck Palahniuk : Invisible Monsters (1999)

Axioms of a Vector Space

- Let V be a nonempty set of elements $\mathbf{a}, \mathbf{b}, \mathbf{c}$, which are called vectors. We say that V is a vector space if there are defined two algebraic operations, vector addition and scalar multiplication, such that,
  - Commutativity : $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
  - Associativity : $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
  - Additive Identity : $\mathbf{a} + \mathbf{0} = \mathbf{a}$ for every $\mathbf{a} \in V$
  - Additive Inverse : $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$ for every $\mathbf{a} \in V$
  - Distributivity : Let $\alpha, \beta \in \mathbb{R}$ then $(\alpha + \beta)(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{a} + \alpha\mathbf{b} + \beta\mathbf{a} + \beta\mathbf{b}$
  - Associativity : Let $\alpha, \beta \in \mathbb{R}$ then $\alpha(\beta\mathbf{a}) = (\alpha\beta)\mathbf{a}$
  - Multiplicative Identity : $1\mathbf{a} = \mathbf{a}$
- $\mathbb{R}^n$ is a vector space.

Examples of Abstract Vector Spaces

- The space of n^{th}-order polynomials, $\mathbb{P}_n$.
- The space of continuous functions on the real-line, $C(\mathbb{R})$.
- The space of n-times continuously differentiable functions on the real line, $C^n(\mathbb{R})$.
- The space of periodic functions whose principle period is from $-\pi$ to π.
- The space of square integrable functions on the domain (a, b), $L^2\,(a, b)$.

Fundamental Questions - Part I

In linear algebra we learned how to define a vector space by teasing out a basis set of linearly independent vectors via row-reduction. This constructive approach illustrates the concept of a vector space but the question is more often phrased in the following way:

- Given a vector space V how can one represent any element in this space?

Answer: Use a linear combination a basis vectors. This leads to another question:

- For abstract spaces how does one find a set of basis vectors?

Answer: In the abstract setting more math is required. Lucky for us this is the job of a mathematician and bases of common abstract spaces are already defined.

Fundamental Questions - Part II

Question: Assuming that we have a basis for a space then how do we find the coefficients in the linear combination required to represent some element of the space?

- Answer One: If there are a finite number of basis vectors then the problem of finding coefficients $c_1, c_2, c_2, \ldots, c_n$ for,

$$\mathbf{y} = \sum_{i=j}^{n} c_j \mathbf{v}_j, \tag{1}$$

is a problem of row-reduction. This is true regardless of the space since any vector space of dimension n is isomorphic to $\mathbb{R}^n$. If there are an infinite number of basis vectors then the problem is more difficult.

Fundamental Questions - Part III

- <u>Answer Two</u>: In the case of an infinite dimensional vector space the the linear combination reads,

$$\mathbf{y} = \sum_{i=j}^{\infty} c_j \mathbf{v}_j, \tag{2}$$

and the iterative process of row-reduction cannot hope to find the infinite number of coefficients. In this case we must find other approaches. These techniques are more powerful but require geometric tools.

Abstract Inner-Product Spaces

A vector space is defined by its cardinal directions but has no geometric meaning. A vector space V equipped with a product $\langle \cdot, \cdot \rangle$ such that:

- Linearity : Let $\alpha, \beta \in \mathbb{R}$ then $\langle \alpha \mathbf{a} + \beta \mathbf{b}, \mathbf{c} \rangle = \alpha \langle \mathbf{a}, \mathbf{c} \rangle + \beta \langle \mathbf{bc} \rangle$.
- Symmetry : $\langle \mathbf{a}, \mathbf{b} \rangle = \langle \mathbf{b}, \mathbf{a} \rangle$
- Positive-definiteness : $\langle \mathbf{a}, \mathbf{a} \rangle \geq 0$ and $\langle \mathbf{a}, \mathbf{a} \rangle = 0 \iff \mathbf{a} = 0$

This additional structure induces the following properties:

- Orthogonality : $\langle \mathbf{a}, \mathbf{b} \rangle = 0$ implies that **a** is orthogonal to **b**
- Length/Norm : $||\mathbf{a}|| = \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle}$.
- Cauchy-Schwarz :$| \langle \mathbf{a}, \mathbf{b} \rangle | \leq ||\mathbf{a}|| \, ||\mathbf{b}||$
- Parallelogram Rule : $||\mathbf{a} + \mathbf{b}||^2 + ||\mathbf{a} - \mathbf{b}||^2 = 2(||\mathbf{a}||^2 + ||\mathbf{b}||^2)$
- Triangle Inequality : $||\mathbf{a} + \mathbf{b}|| \leq ||\mathbf{a}|| + ||\mathbf{b}||$

Inner-Products and Orthogonal Expansions

Suppose that the vector space V with basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots\}$ is equipped with an inner-product $\langle \cdot, \cdot \rangle$ such that $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \delta_{ij}$. Then for any $\mathbf{y} \in V$ we have,

$$\mathbf{y} = \sum_{i=j}^{\infty} c_j \mathbf{v}_j \iff \langle \mathbf{v}_i, \mathbf{y} \rangle = \left\langle \mathbf{v}_i, \sum_{j=1}^{\infty} c_j \mathbf{v}_j \right\rangle \tag{3}$$

$$= \sum_{j=1}^{\infty} c_j \langle \mathbf{v}_i, \mathbf{v}_j \rangle \tag{4}$$

$$= \sum_{j=1}^{\infty} c_j \delta_{ji} = c_i, \tag{5}$$

which implies that the i^{th} coefficient is given by $c_i = \langle \mathbf{v}_i, \mathbf{y} \rangle$ for $i = 1, 2, 3, \ldots$.

The Abstract Inner-Product Space : $L_2\left(-\pi, \pi\right)$

We define $L_2\left(-\pi, \pi\right)$ to be the space of functions such that $\displaystyle\int_{-\pi}^{\pi} \left(f\left(x\right)\right)^2 dx < \infty$. In homework 2 problem 5 we showed that $\displaystyle\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$ satisfies the axioms of inner-product. Consequently, we now have that,

- $L_2(-\pi, \pi)$ is a vector-space of functions which have finite length with respect to the inner-product $\displaystyle\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$.

This is all well and good but we still have an important question:

- How do we represent elements in this space?

The Concept of Periodic Extension

First, we should decide how to think about elements in the space $L_2(-\pi, \pi)$. Right now all we know is that elements in this space must have the property that,

- If $f \in L_2(-\pi, \pi)$ then the area under the curve of its square from $-\pi$ to π is finite.

This is not telling us much. However, have the following thought.

- The function f is not periodic but if we repeatedly copy its shape to the space outside of $(-\pi, \pi)$ then this extension, f^* <u>is</u> periodic with principle period of 2π.

We call this new function f^* the periodic extension of f with the property that $f^*(x) = f(x)$ for all $x \in (-\pi, \pi)$. Now, our goal is to search for a representation of f^* knowing that we will get $f \in L_2(-\pi, \pi)$ for free.

The Fourier Basis

Forgetting about $L_2(-\pi, \pi)$ for a second we ask the following question.

- Suppose that we have an arbitrary 2π-periodic function. What is a basis for the space of 2π-periodic functions?

Consider the following set of functions,

$$\{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \cos(3x), \sin(3x), \ldots\}, \tag{6}$$

which is called the Fourier basis. It can be shown these functions serve as a basis for all 2π-periodic functions with finite integral on their principle period. Thus, we now hope that,

$$f^*(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx). \tag{7}$$

The Main Result

We now have the problem of finding the coefficients a_0, a_n, b_n. Lucky for us the Fourier basis is an orthogonal basis. Using the orthogonality of the basis vectors we have our final result,

$$f \in L_2(-\pi, \pi) \Rightarrow f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx), \tag{8}$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \tag{9}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \tag{10}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx, \tag{11}$$

which are sometimes called the Euler formulas.