Partial Differential Equations : Heat Equation, Wave Equation, Properties, External Forcing

Text: 12.3-12.5
Lecture Notes : 14 and 15
Lecture Slides: 6

```
Quote of Homework Eight
```

Arrakis teaches the attitude of the knife chopping off what's incomplete and saying: "Now it's complete because it's ended here."

Frank Herbert : Dune (1965)

## 1. Heat Equation on a closed and bounded spatial domain of $\mathbb{R}^{1+1}$

Consider the one-dimensional heat equation,

$$
\begin{gather*}
\frac{\partial u}{\partial t}=c^{2} \frac{\partial^{2} u}{\partial x^{2}} \quad,  \tag{1}\\
x \in(0, L), \quad t \in(0, \infty), \quad c^{2}=\frac{K}{\sigma \rho} . \tag{2}
\end{gather*}
$$

Equations (1)-(2) model the time-evolution of the temperature, $u=u(x, t)$, of a heat conducting medium in one-dimension. The object, of length $L$, is assumed to have a homogenous thermal conductivity $K$, specific heat $\sigma$, and linear density $\rho$. That is, $K, \sigma, \rho \in \mathbb{R}^{+}$. If we consider an object of finite-length, positioned on say $(0, L)$, then we must also specify the boundary conditions ${ }^{1}$,

$$
\begin{equation*}
u_{x}(0, t)=0, u_{x}(L, t)=0, \tag{3}
\end{equation*}
$$

Lastly, for the problem to admit a unique solution we must know the initial configuration of the temperature,

$$
\begin{equation*}
u(x, 0)=f(x) \tag{4}
\end{equation*}
$$

1.1. Separation of Variables: General Solution. Assume that the solution to (1)-(2) is such that $u(x, t)=F(x) G(t)$ and use separation of variables to find the general solution to (1)-(2), which satisfies (3)-(4). ${ }^{2}$
1.2. Qualitative Dynamics. Describe how the long term behavior of the general solution to (1)-(4) changes as the thermal conductivity, $K$, is increased while all other parameters are held constant. Also, describe how the solution changes when the linear density, $\rho$, is increased while all other parameters are held constant.
1.3. Fourier Series : Solution to the IVP. Define,

$$
f(x)=\left\{\begin{array}{cc}
\frac{2 k}{L} x, & 0<x \leq \frac{L}{2}  \tag{5}\\
\frac{2 k}{L}(L-x), & \frac{L}{2}<x<L
\end{array}\right.
$$

and for the following questions we consider the solution, $u$, to the heat equation given by, (1)-(2), which satisfies the initial condition given by (11). ${ }^{3}$ For $L=1$ and $k=1$, find the particular solution to (1)-(2) with boundary conditions (3)-(4) for when the initial temperature profile of the medium is given by (11). Show that $\lim _{t \rightarrow \infty} u(x, t)=f_{\text {avg }}=0.5{ }^{4}$

[^0]Consider the one-dimensional wave equation,

$$
\begin{align*}
\frac{\partial^{2} u}{\partial t^{2}}=c^{2} \frac{\partial^{2} u}{\partial x^{2}}  \tag{6}\\
x \in(0, L), \quad t \in(0, \infty), \quad c^{2}=\frac{T}{\rho} . \tag{7}
\end{align*}
$$

Equations (1)-(2) model the time-evolution of the displacement from rest, $u=u(x, t)$, of an elastic medium in one-dimension. The object, of length $L$, is assumed to have a homogeneous lateral tension $T$, and linear density $\rho$. That is, $T, \rho \in \mathbb{R}^{+}$. Assume, as well, the boundary conditions ${ }^{5}$,

$$
\begin{equation*}
u_{x}(0, t)=0, u_{x}(L, t)=0, \tag{8}
\end{equation*}
$$

and initial conditions,

$$
\begin{gather*}
u(x, 0)=f(x),  \tag{9}\\
u_{t}(x, 0)=g(x) . \tag{10}
\end{gather*}
$$

2.1. Separation of Variables : General Solution. Assume that the solution to (6)-(7) is such that $u(x, t)=F(x) G(t)$ and use separation of variables to find the general solution to (6)-(7), which satisfies (8)-(10). ${ }^{6}{ }^{7}$
2.2. Qualitative Dynamics. Describe how the the general solution to (6)-(7) changes as the tension, $T$, is increased while all other parameters are held constant. Also, describe how the solution changes when the linear density, $\rho$, is increased while all other parameters are held constant.

### 2.3. Fourier Series : Solution to the IVP. Define,

$$
f(x)=\left\{\begin{array}{cc}
\frac{2 k}{L} x, & 0<x \leq \frac{L}{2}  \tag{11}\\
\frac{2 k}{L}(L-x), & \frac{L}{2}<x<L
\end{array}\right.
$$

Let $L=1$ and $k=1$ and find the particular solution, which satisfies the initial displacement, $f(x)$, given by (11) and has zero initial velocity for all points on the object.

## 3. D'alembert Solution to the Wave Equation in $\mathbb{R}^{1+1}$

Show that by direct substitution that the function $u(x, t)$ given by,

$$
\begin{equation*}
u(x, t)=\frac{1}{2}\left[u_{0}(x-c t)+u_{0}(x+c t)\right]+\frac{1}{2 c} \int_{x-c t}^{x+c t} v_{0}(y) d y \tag{12}
\end{equation*}
$$

is a solution to the one-dimensional wave equation where $u_{0}$ and $v_{0}$ are the ideally elastic objects initial displacement and velocity, respectively. ${ }^{8}$

[^1]It makes sense to consider time-dependent interface conditions. That is, (1) and (4) subject to

$$
\begin{equation*}
u(0, t)=g(t), \quad u(L, t)=h(t), \quad t \in(0, \infty) \tag{13}
\end{equation*}
$$

Show that this PDE transforms into:

$$
\begin{array}{cl}
\frac{\partial w}{\partial t}=c^{2} \frac{\partial^{2} w}{\partial x^{2}}-S_{t}(x, t) & , \\
x \in(0, L), & t \in(0, \infty), \tag{15}
\end{array} c^{2}=\frac{\kappa}{\rho \sigma} .
$$

with boundary conditions and initial conditions,

$$
\begin{array}{r}
w(0, t)=w(L, t)=0 \\
w(x, 0)=F(x) \tag{17}
\end{array}
$$

where $F(x)=f(x)-S(x, 0)$ and $S(x, t)=\frac{h(t)+g(t)}{L} x+g(t) .{ }^{9}$

## 5. Inhomogeneous Wave Equation on a closed and bounded spatial domain of $\mathbb{R}^{1+1}$

Consider the non-homogeneous one-dimensional wave equation,

$$
x \in(0, L), \quad \begin{array}{cc}
\frac{\partial^{2} u}{\partial t^{2}}=c^{2} \frac{\partial^{2} u}{\partial x^{2}}+F(x, t) & \\
t \in(0, \infty), & c^{2}=\frac{T}{\rho} . \tag{19}
\end{array}
$$

with boundary conditions and initial conditions,

$$
\begin{array}{r}
u(0, t)=u(L, t)=0 \\
u(x, 0)=u_{t}(x, 0)=0 \tag{21}
\end{array}
$$

Letting $F(x, t)=A \sin (\omega t)$ gives the following Fourier Series Representation of the forcing function $F$,

$$
\begin{equation*}
F(x, t)=\sum_{n=1}^{\infty} f_{n}(t) \sin \left(\frac{n \pi x}{L}\right) \tag{22}
\end{equation*}
$$

where

$$
\begin{equation*}
f_{n}(t)=\frac{2 A}{n \pi}\left(1-(-1)^{n}\right) \sin (\omega t) \tag{23}
\end{equation*}
$$

5.1. Educated Fourier Series Guessing. Based on the boundary conditions we assume a Fourier sine series solution. However, the time-dependence is unclear. So, assume that,

$$
\begin{equation*}
u(x, t)=\sum_{n=1}^{\infty} \sin \left(\frac{n \pi}{L} x\right) G_{n}(t) \tag{24}
\end{equation*}
$$

where $G_{n}(t)$ represents the unknown dynamics of the $n$-th Fourier mode. Using this assumption and (22)-(23), show by direct substitution that (18) yields the ODE,

$$
\begin{equation*}
\ddot{G}_{n}+\left(\frac{c n \pi}{L}\right)^{2} G_{n}=\frac{2 A}{n \pi}\left(1-(-1)^{n}\right) \sin (\omega t) . \tag{25}
\end{equation*}
$$

5.2. Solving for the Dynamics. The solution to (25) is given by,

$$
\begin{equation*}
G_{n}(t)=G_{n}^{h}(t)+G_{n}^{p}(t) \tag{26}
\end{equation*}
$$

where $G_{n}^{h}(t)=B_{n} \cos \left(\frac{c n \pi}{L} t\right)+B_{n}^{*} \sin \left(\frac{c n \pi}{L} t\right)$ is the homogeneous solution and $G_{n}^{p}(t)$ is the particular solution to (25).
5.2.1. Particular Solution - I. If $\omega \neq c n \pi / L$ then what would the choice for $G_{n}^{p}(t)$ be, assuming you were solving for $G_{n}^{p}(t)$ using the method of undetermined coefficients? DO NOT SOLVE FOR THESE COEFFICIENTS
5.2.2. Particular Solution - II. If $\omega=c n \pi / L$ then what would the choice for $G_{n}^{p}(t)$ be, assuming you were solving for $G_{n}^{p}(t)$ using the method of undetermined coefficients? DO NOT SOLVE FOR THESE COEFFICIENTS
5.2.3. Physical Conclusions. For the Particular Solution - II, what is $\lim _{t \rightarrow \infty} u(x, t)$ and what does this limit imply physically?

[^2]
[^0]:    ${ }^{1}$ Here the boundary conditions correspond to perfect insulation of both endpoints
    ${ }^{2}$ An insulated bar is discussed in examples 4 and 5 on page 557 .
    ${ }^{3}$ When solving the following problems it would be a good idea to go back through your notes and the homework looking for similar calculations.
    ${ }^{4}$ It is interesting here to note that though the initial condition $f$ doesn't appear to satisfy the boundary conditions its periodic Fourier extension does. That is, if you draw the even periodic extension of the initial condition then you would see that the slope is not well defined at the end points. Remembering that the Fourier series averages the right and left hand limits of the periodic extension of the function $f$ at the endpoints shows that the boundary conditions are, in fact, satisfied, since the derivative of an average is the average of derivatives.

[^1]:    ${ }^{5}$ These boundary conditions imply that the object must have zero slope at each endpoint.
    ${ }^{6}$ It is important to notice that the solution to the spatial portion of the problem is the same as the heat problem above.
    ${ }^{7}$ Remember that in this case we have a nontrivial spatial solution for zero eigenvalue. From this you should find the associated temporal function should find that $G_{0}(t)=C_{1}+C_{2} t$.
    ${ }^{8}$ This is called the D'Alembert solution to the wave equation. To do this you may want to recall the fundamental theorem of calculus, $\frac{d}{d x} \int_{0}^{x} f(t) d t=$ $f(x)$ and properties of integrals, $\int_{-a}^{a} f(x) d x=\int_{-a}^{0} f(x) d x+\int_{0}^{a} f(x) d x$.

[^2]:    ${ }^{9} \mathrm{~A}$ similar transformation can be found for the wave equation with inhomogeneous boundary conditions. The moral here is that time-dependent boundary conditions can be transformed into externally driven (AKA Forced or inhomogeneous) PDE with standard boundary conditions.

