MATH 332 - Linear Algebra
Homework 5, Summer 2009

July 2, 2009
Due: July 6, 2009

Determinants - Row-Reductions - Properties - Inverse Matrices - Volumes

1. Given the following for matrices:

$$\mathbf{A}=\begin{bmatrix} a & b \\ c & d \end{bmatrix},\quad \mathbf{B}=\begin{bmatrix} c & d \\ a & b \end{bmatrix},\quad \mathbf{C}=\begin{bmatrix} a & b \\ kc & kd \end{bmatrix},\quad \mathbf{D}=\begin{bmatrix} a+kc & b+kd \\ c & d \end{bmatrix}.$$

Calculate the determinants of the previous matrices by theorem 2.2.4. In each case, state the row-operation on $\mathbf{A}$ and describe how it effects the determinant.

2. The following questions illustrate some important properties of the determinant.

(a) The determinant is not, in general, a linear mapping. That is, det: $\mathbb{R}^{n\times n}\to\mathbb{R}$ is **not**, in general, such that, $\det(\mathbf{A}+\mathbf{B})=\det(\mathbf{A})+\det(\mathbf{B})$. The determinant is, in general, *multilinear*.[1] Show this for the domain $\mathbb{R}^{3\times 3}$ by verifying that $\det(\mathbf{A})=\det(\mathbf{B})+\det(\mathbf{C})$, where $\mathbf{A},\mathbf{B},\mathbf{C}$ are given as,

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} & u_1+v_1 \\ a_{21} & a_{22} & u_2+v_2 \\ a_{31} & a_{32} & u_3+v_3 \end{bmatrix},\quad \mathbf{B}=\begin{bmatrix} a_{11} & a_{12} & u_1 \\ a_{21} & a_{22} & u_2 \\ a_{31} & a_{32} & u_3 \end{bmatrix},\quad \mathbf{C}=\begin{bmatrix} a_{11} & a_{12} & v_1 \\ a_{21} & a_{22} & v_2 \\ a_{31} & a_{32} & v_3 \end{bmatrix}.$$

(b) Show that if $\mathbf{A}$ is invertible, then $\det(\mathbf{A}^{-1})=\frac{1}{\det(\mathbf{A})}$.

(c) Let $\mathbf{A}$ and $\mathbf{P}$ be square matrices such that $\mathbf{P}^{-1}$ exists. Show that $\det(\mathbf{PAP}^{-1})=\det(\mathbf{A})$.

(d) Let $\mathbf{U}$ be a square matrix such that $\mathbf{U}^T\mathbf{U}=\mathbf{I}$. Show that $\det(\mathbf{U})=\pm 1$.

(e) Find a formula for $\det(r\mathbf{A})$ where $\mathbf{A}\in\mathbb{R}n\times n$ and $r\in\mathbb{R}$.

3. Given,

$$\mathbf{A}=\begin{bmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{bmatrix}. \tag{1}$$

(a) Find $\det(\mathbf{A})$ using cofactor expansion.

(b) Find $\det(\mathbf{A})$ using row reduction to echelon form.

4. Given,

$$\mathbf{A}=\begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}.$$

Calculate the adjugate of $\mathbf{A}$ and using theorem 3.3.8 calculate $\mathbf{A}^{-1}$.

5. The determinant has a geometric interpretation. In $\mathbb{R}^2$, $\det(\mathbf{A})$ is the area of the parallelogram formed by the two vectors $\mathbf{a}_1,\mathbf{a}_2$, where $\mathbf{A}=[\mathbf{a}_1\,\mathbf{a}_2]$. In $\mathbb{R}^3$, $\det(\mathbf{A})$ is the volume of the parallelepiped formed by the three vectors $\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3$, where $\mathbf{A}=[\mathbf{a}_1\,\mathbf{a}_2\,\mathbf{a}_3]$.

Using the concept of volume, explain why the determinant of a 3×3 matrix $\mathbf{A}$ is zero if and only if $\mathbf{A}$ is not invertable.

Note: No credit will be given for the use of Theorem 3.2.4. Also, note that there are two proofs here. The forward proof should assume that $\det(\mathbf{A})=0$ and conclude that $\mathbf{A}$ is singular by using the geometry formed by the column vectors. The backward proof should start assuming $\mathbf{A}$ is singular and conclude that the parallelepiped volume is zero. The two proofs together prove the *if and only if* statement above.

Hint: Use the invertible matrix theorem of 2.3 and a geometric description of linearly dependent vectors in $\mathbb{R}^3$.

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[1] A multilinear map is a mathematical function of several vector variables that is linear in each variable. That is, if all columns except one are fixed, then the determinant is a linear function of that one column. See `http://en.wikipedia.org/wiki/Multilinear_map` for more information.

Homework 5 - Solutions

1. i. $\det(A) = ad - bc$

ii. $\det(B) = cb - ad = -(ad - bc) = -\det(A)$

iii. $\det(D) = d(a+kc) - c(b+kd) = ad + kdc - cb - cd \cdot k =$

Note. ↑ I switched

$= ad - bc = \det(A)$

iv. $\det(C) = ad \cdot k - kc \cdot b = k(ad - bc) = k\det(A)$

$A \sim B$ by a row interchange and ii. shows $\det(A) = -\det(B)$
$A \sim C$ by a Row Scaling and iv. shows $\det(B) = k\det(A)$
$A \sim D$ by a Row interchange where a multiple of 1 Row is added to another. iii. shows that $\det(A) = \det(D)$.

2. a. Elementary matrices E_5, E_6 corresponds to Row interchanges.

$\det(E_5) = -1 \cdot \det\begin{pmatrix}1&0\\0&1\end{pmatrix} = (-1)\cdot 1 = -1$

$\det(E_6) = 1 \cdot \det\begin{pmatrix}0&1\\1&0\end{pmatrix} = -1$

The determinant of a elementary Row Interchange matrix is -1.

b. Elementary matrices E_3, E_4 correspond to Row scaling.

$\det(E_3) = k \cdot 1 \cdot 1 = k$ by diagonal matrix theorem.

$\det(E_4) = 1 \cdot k \cdot 1 = k$

The determinate of a Elementary Row Scaling by k is k.

c. Elementry matrices E_1, E_2 correspond to Row Replacements.

$\det(E_1) = 1 \cdot \det\begin{pmatrix}1&0\\k&1\end{pmatrix} = 1\cdot(1-0) = 1$

$\det(E_2) = 1 \cdot \det\begin{pmatrix}1&0\\k&1\end{pmatrix} = 1$

the determinant of an elementary Row Replacement is 1.

Homework 6 - Solutions

1.

a.

(1) $$\det(A) = a_{11} \cdot \det\begin{pmatrix} a_{22} & u_2+v_2 \\ a_{32} & u_3+v_3 \end{pmatrix} - a_{12} \det\begin{pmatrix} a_{21} & u_2+v_2 \\ a_{31} & u_3+v_3 \end{pmatrix} +$$

$$+ (u_1+v_1) \det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

(2) $$\det(B) = a_{11} \det\begin{pmatrix} a_{22} & u_2 \\ a_{32} & u_3 \end{pmatrix} + -a_{12} \det\begin{pmatrix} a_{21} & u_2 \\ a_{31} & u_3 \end{pmatrix} + u_1 \cdot \det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

(3) $$\det(C) = a_{11} \cdot \det\begin{pmatrix} a_{22} & v_2 \\ a_{32} & v_3 \end{pmatrix} - a_{12} \det\begin{pmatrix} a_{21} & v_2 \\ a_{31} & v_3 \end{pmatrix} + v_1 \cdot \det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

Since,

(4) $$a_{11}\left\{\det\begin{pmatrix} a_{22} & u_2 \\ a_{32} & u_3 \end{pmatrix} + \det\begin{pmatrix} a_{22} & v_2 \\ a_{32} & v_3 \end{pmatrix}\right\} = a_{11}\{a_{22}u_3 - a_{32}u_2 + a_{22}v_3 - a_{32}v_2\}$$

(5) $$a_{11} \det\begin{pmatrix} a_{22} & u_2+v_2 \\ a_{32} & u_3+v_3 \end{pmatrix} = a_{11} \cdot \{a_{22}(u_3+v_3) - a_{22}(u_2+v_2)\}$$

(6) $$-a_{12}\left\{\det\begin{pmatrix} a_{21} & u_2 \\ a_{31} & u_3 \end{pmatrix} + \det\begin{pmatrix} a_{21} & v_2 \\ a_{31} & v_3 \end{pmatrix}\right\} = -a_{12} \cdot \{a_{21} \cdot u_3 - a_{31}u_2 +$$

$$+ a_{21}v_3 - a_{31}v_2\}$$

(7) $$-a_{12} \det\begin{pmatrix} a_{21} & v_2+u_2 \\ a_{31} & u_3+v_3 \end{pmatrix} = -a_{12}\{a_{21}(u_3+v_3) - a_{31}(v_2+u_2)\}$$

(4) = (5), (6) = (7) $\Rightarrow$ $\det(A) = \det(B) + \det(C)$.

b. $\det(AA^{-1}) = \det(A)\cdot\det(A^{-1}) = \det(I) \iff$ (By theorem

$\iff \det(A)\cdot\det(A^{-1}) = 1 \iff$ (By theorem

$$\iff \det(A^{-1}) = \frac{1}{\det(A)}$$

$\boxtimes$

c. $\det(PAP^{-1}) = \det(P)\cdot\det(A)\cdot\det(P^{-1}) =$ (By theorem

$= \det(P)\dfrac{1}{\det(P)}\cdot\det(A) = \det(A)$ (By commutivity of scalars + part b.

d. $\det(u^T u) = \det(u^T)\det(u) =$ (By theorem

$= (\det(u))^2 = \det(I) = 1 \iff$

$\iff \sqrt{(\det(u))^2} = \pm\det(u) = 1 \Rightarrow \det(u) = \pm 1$ $\boxtimes$

e) If $A \in \mathbb{R}^{n\times n}$ then rA is a matrix that has n-many Row scalings

~~det~~ $= \det(rA) = r^n \det(A)$

2. $$\det(A) = \begin{vmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 3 & 0 & 4 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 3 & 0 & 4 \\ 0 & 2 & 1 \\ 2 & 0 & 5/2 \end{vmatrix} =$$

$$= 2 \cdot \det\begin{pmatrix} 3 & 4 \\ 2 & 5/2 \end{pmatrix} = 2 \cdot \left\{ \frac{3 \cdot 5}{2} - 2 \cdot 4 \right\} = 2 \cdot (-1/2) = -1$$

$$\operatorname{adj} A = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

where

$$C_{11} = \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 8 - 3 = 5, \quad C_{21} = -\begin{vmatrix} 6 & 7 \\ 3 & 4 \end{vmatrix} = -(24-21) = -3, \quad C_{31} = \begin{vmatrix} 6 & 7 \\ 2 & 1 \end{vmatrix} = 6 - 14 = -8$$

$$C_{12} = -\begin{vmatrix} 0 & 1 \\ 2 & 4 \end{vmatrix} = +2, \quad C_{22} = \begin{vmatrix} 3 & 7 \\ 2 & 4 \end{vmatrix} = 12 - 14 = -2, \quad C_{32} = -\begin{vmatrix} 3 & 7 \\ 0 & 1 \end{vmatrix} = -3$$

$$C_{13} = \begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix} = -4, \quad C_{23} = -\begin{vmatrix} 3 & 6 \\ 2 & 3 \end{vmatrix} = -(9-12) = 3, \quad C_{33} = \begin{vmatrix} 3 & 6 \\ 0 & 2 \end{vmatrix} = 6 - 0 = 6$$

Using theorem 3.38 we have that

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A) = -1 \cdot \begin{bmatrix} 5 & -3 & -8 \\ 2 & -2 & -3 \\ -4 & 3 & 6 \end{bmatrix} = \begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix} = A^{-1}$$

3) a

$$\det(A) = 1\cdot\det\begin{pmatrix}-3 & 3\\ 13 & -7\end{pmatrix} - 5\det\begin{pmatrix}3 & 3\\ 2 & -7\end{pmatrix} + -3\det\begin{pmatrix}3 & -3\\ 2 & 13\end{pmatrix} =$$

$$= -18 - 5\cdot -27 - 3\cdot 45 = -18$$

b)

$$\begin{vmatrix}1 & 5 & -3\\ 3 & -3 & 3\\ 2 & 13 & -7\end{vmatrix} = \begin{vmatrix}1 & 5 & -3\\ 0 & -18 & 12\\ 0 & 3 & -1\end{vmatrix} =$$

$$= +6\begin{vmatrix}1 & 5 & -3\\ 0 & -3 & 2\\ 0 & +3 & -1\end{vmatrix} = 6\begin{vmatrix}1 & 5 & -3\\ 0 & -3 & 2\\ 0 & 0 & 1\end{vmatrix} = 6\cdot -3 = -18$$

5.

Forward direction-

Assume $A_{3\times3}$ is such that $\det(A) = 0$. Then the volume of the parallepiped spanned by $\vec{a}_1, \vec{a}_2, \vec{a}_3$ has zero volume. That is, the parallepiped does not exist. This implies that all the vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ lie in the same plane, and form a linearly dependent set. Thus, by the invertable matrix theorem A^{-1} does not exist.

Backward direction:

Assume A is not invertable. Then the columns of A are linearly dependent and $\text{span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ is at most (in terms of dimension) a plane which has zero volume and cannot form a parallepiped. Thus, $\det(A) = 0$.

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