or simply

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0.} \qquad (60.5c)$$

This is a **partial differential equation** (an equation involving partial derivatives). It expresses a relationship between traffic density and traffic flow derived by assuming that the number of cars is conserved, that is, cars are not created nor destroyed. It is valid everywhere (all x) and for all time. It is called the equation of **conservation of cars**.

(2) The equation of conservation of cars can be derived more expeditiously. Consider the intergral conservation law, equation 60.4a, for any finite segment of highway, $a \leq x \leq b$. Now take the partial derivative with respect to b. (This is equivalent to letting $b = a + \Delta a$, dividing by Δa, and taking the limit as $\Delta a \longrightarrow 0$). Thus,

$$\frac{\partial \rho(b, t)}{\partial t} = -\frac{\partial}{\partial b}(q(b, t)).$$

Since b represents any position on the road, again b is replaced by x, yielding the equation of conservation of cars, equation 60.5.

(3) An alternate derivation for a roadway of finite length ($a \leq x \leq b$) is based on noting that the following relation is clearly valid for the right-hand side of equation 60.4a,

$$q(a, t) - q(b, t) = -\int_a^b \frac{\partial}{\partial x}[q(x, t)]\, dx.$$

Thus, from equation 60.4a,

$$\int_a^b \left[\frac{\partial \rho(x, t)}{\partial t} + \frac{\partial q(x, t)}{\partial x}\right] dx = 0. \qquad (60.6)$$

Equation 60.5 immediately follows by taking the derivative with respect to b, as was done in derivation (2). However, let us briefly discuss a different and powerful argument. Equation 60.6 states that the definite integral of some quantity is always zero for all values of the independently varying limits of the integral. The only function whose integral is zero for *all* intervals is the zero function. Hence, again equation 60.5 follows.

By three equivalent methods, we have shown that

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0} \qquad (60.7)$$

must be valid if there are no entrances nor exits along a roadway, expressing

the conservation of cars. Equation 60.7 is valid in many situations having nothing to do with traffic. In general, if ρ is any local quantity that is conserved (in one spatial dimension) and if q is the flow of that quantity across a boundary (often in physics called the **flux**), then it can be shown, using the same arguments we have just developed, that equation 60.7 is valid!* Sometimes experiments must be performed to determine how the flow q depends on other quantities in the problem. However, for traffic problems, we know from Sec. 59 that

$$\boxed{q = \rho u,}$$

and thus conservation of cars can be written as

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0,} \qquad (60.8)$$

a partial differential equation relating the traffic density and the velocity field.

EXERCISES

60.1. Consider a semi-infinite highway $0 \leq x < \infty$ (with no entrances or exits other than at $x = 0$). Show that the number of cars on the highway at time t is

$$N_0 + \int_0^t q(0, \tau)\, d\tau,$$

where N_0 is the number of cars on the highway at $t = 0$. (You may assume that $\rho(x, t) \longrightarrow 0$ as $x \longrightarrow \infty$.)

60.2. Suppose that we are interested in the change in the number of cars $N(t)$ between two observers, one fixed at $x = a$ and the other moving in some prescribed manner, $x = b(t)$:

$$N(t) = \int_a^{b(t)} \rho(x, t)\, dx$$

(a) The derivative of an integral with a variable limit is

$$\frac{dN}{dt} = \frac{db}{dt}\rho(b, t) + \int_a^{b(t)} \frac{\partial \rho}{\partial t}\, dx.$$

(Note that the integrand, $\rho(x, t)$, also depends on t.) Show this result

*In two or three dimensions using the divergence theorem equation 60.7 must be replaced by $\partial\rho/\partial t + \nabla\cdot\vec{q} = 0$, where $\vec{q}$ is the vector flow of the quantity whose density is ρ.