4 _ 11 _ 08

Note Title 4/9/2008

Since L^2 commutes with each component we can say

$$[L^2, \vec{L}] = 0$$

Since L^2 is compatible (commutes) with each component of L, lets pick one and solve the simultaneous Σ-value problem:

$$\underbrace{L^2 f = \lambda f \qquad L_z f = \mu f}$$

we can only do this for observables that commute.

we'll do this by the algebraic method. i.e ladder operators.

$$L_\pm = L_x \pm i L_y$$

Now can show that

$$[L_z, L_\pm] = \pm \hbar L_\pm$$

$$[L^2, L_\pm] = 0$$

$$[L_z, L_\pm] = [L_z, L_x \pm i L_y]$$
$$= [L_z, L_x] \pm i [L_z, L_y]$$
$$= i\hbar L_y \pm i(-i\hbar L_x)$$
$$= \hbar(\pm L_x + i L_y)$$
$$= \pm\hbar(L_x \pm i L_y) = \pm\hbar L_\pm$$

So lets look at

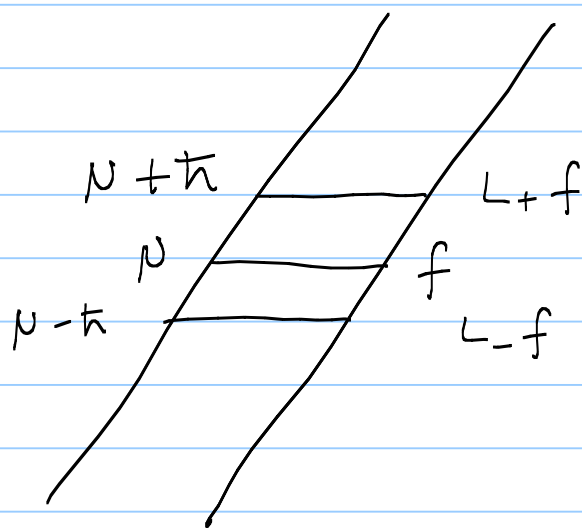
$$L_z(L_\pm f)$$ The arg. goes just like it did for the harm. oscillator

$$L_z L_\pm f = \underbrace{\underbrace{L_z L_\pm f - L_\pm L_z f}_{[L_z, L_\pm] f}}_{\pm\hbar f L_\pm} + L_\pm \underbrace{L_z f}_{\mu f}$$

So $\boxed{L_z(L_\pm f) = (\mu \pm \hbar)(L_\pm f)}$

L_+ is the raising operator for angular momentum.

L_- is the lowering operator



eventually this must terminate or else we'll reach a state for which L_z exceeds $|L|$ which cannot happen. Hence

there exists a state f_t such that

$$L_+ f_t = 0$$

So $\boxed{L_z f_t = \hbar l f_t}$ for some l

Now lets see what L^2 does. we have assumed that

$L^2 f_t = \lambda f_t$ now Lets comp. λ.

Somehow, if we can express L^2 in terms of what we already know (L_z and $L_\pm$) then we'll be OK.

$$\boxed{\text{show} \qquad L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z}$$

Then $L^2 f_t =$

$$(L_- L_+ + \underbrace{L_z^2}_{\hbar^2 \ell^2 f_t} + \underbrace{\hbar L_z) f_t}_{\hbar^2 \ell f_t} \qquad L_+ f_t = 0$$

$$= \hbar^2 \ell(\ell+1) f_t$$

So $\boxed{L^2 f_t = \hbar^2 \ell(\ell+1) f_t}$

Similarly by applying L_- we must reach a bottom state

$$L_- f_b = 0 \qquad \text{for some } \bar{\ell}$$

so $$L_z f_b = \hbar \bar{\ell} f_b$$

going through the same arg. as for $L^2 f_t$

we can show $\boxed{L^2 f_b = \hbar^2 \bar{\ell}(\bar{\ell}-1) f_b}$

$$\boxed{L^2 f_t = \hbar^2 \ell(\ell+1) f_t}$$

in going up and down the ladder we're chang the E-values of L_z not L^2, so

$$\hbar^2 \bar{\ell}(\bar{\ell}-1) = \hbar^2 \ell(\ell+1)$$ 🚩

so either $\bar{\ell} = \ell+1$ impossible for

$$\bar{\ell} = -\ell$$

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Thus we conclude that the E-values of L_z are $m\hbar$ and

$$m = \underbrace{-l, -l+1, \ldots\ l-1, l}_{\text{integer steps.}}$$

$2N = l$ (i.e. $l = -l + N$)

So l itself could be integer or half integer

Summary

The E-values of L_z are $m\hbar$ where $-l \leq m \leq l$ in integer steps.

but l can be integer or half int

$$\boxed{\begin{gathered} L^2 f_l^m = \hbar^2 l(l+1) f_l^m \\ L_z f_l^m = \hbar m f_l^m \\ l = 0, \tfrac{1}{2}, 1, \ldots \qquad m = -l, -l+1, \ldots\ l \end{gathered}}$$

problem 4.18

Since $L_\pm f_\ell^m = A_\ell^m f_\ell^{m\pm 1}$

what is A_ℓ^m if the f_ℓ^m are to be normalized.

Lemma $(L_\pm)^+ = L_\mp$ (show this!)

recall $L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z$

so $L_\pm L_\mp = L^2 - L_z^2 \pm \hbar L_z$

$$\langle f_\ell^m | L_\pm L_\mp | f_\ell^m \rangle = \underbrace{\langle f_\ell^m | L^2 - L_z^2 \pm \hbar L_z | f_\ell^m \rangle}$$

$$\hbar^2 \ell(\ell+1) \langle f_\ell^m | f_\ell^m \rangle - \hbar^2 m^2 \pm \hbar^2 m \langle f_\ell^m | f_\ell^m \rangle$$

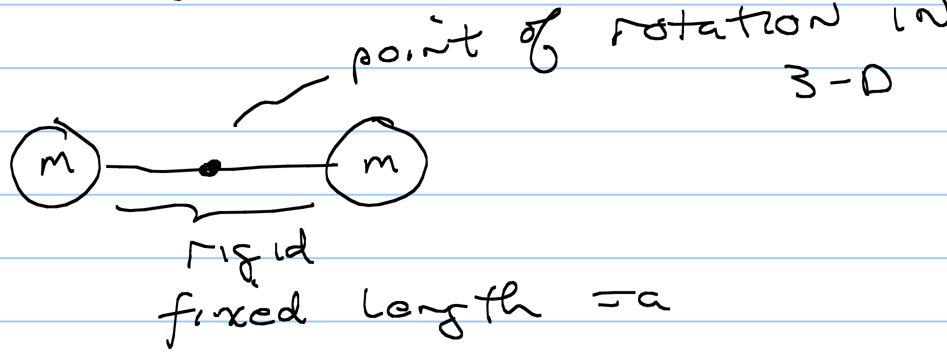
$$\hbar^2 \left[\ell(\ell+1) - m(m \pm 1) \right]$$

$$\langle L_\mp f_\ell^m | L_\mp f_{\ell m} \rangle = \langle A_\ell^m f_\ell^{m\mp 1} | A_\ell^m f_\ell^{m\mp 1} \rangle$$

$$|A_\ell^m|^2$$

$$\Rightarrow \quad |A_\ell^m|^2 = \hbar^2 \left[(\ell(\ell+1) - m(m \pm 1) \right]$$

Problem 4.24



Total energy is 2 x kinetic E of each mass

$$2\left(\tfrac{1}{2} m v^2\right) = m v^2$$

Angular momentum $|\vec{L}| = 2 \frac{a}{2} m v$

$$\Rightarrow L^2 = a^2 m^2 v^2 = a^2 m H$$

$$\Rightarrow \quad H = \frac{L^2}{a^2 m}$$

eigenvalues of L^2 : $\hbar^2 \ell(\ell+1)$

or, since we usually use n to label energies

$$E_n = \frac{1}{a^2 m} \hbar^2 n(n+1)$$

$$\boxed{E_n = \frac{\hbar^2 n(n+1)}{m a^2}}$$

Since $H \propto L^2$ we know that the stationary states are proportional to Y_N^m

degeneracy of the N-th Level is

$2N+1$

Problem $L_+ Y_\ell^\ell = ?$

0

Now $L_z Y_\ell^\ell = \hbar \ell Y_\ell^\ell$

L_z is $-i\hbar \frac{\partial}{\partial\varphi}$

so $-i\cancel{\hbar} \frac{\partial}{\partial\varphi} Y_\ell^\ell = \cancel{\hbar}\ell Y_\ell^\ell$

$$\Rightarrow \boxed{Y_\ell^\ell = f(\theta)\, e^{i\ell\varphi}}$$