Homework #1 Solution:

1.
Given

$$\sigma_1 = \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$
$$\sigma_3 = \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

a)
$$\begin{aligned} \sigma_1^2 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \sigma_2^2 &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} -i^2 & 0 \\ 0 & -i^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \sigma_3^2 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

b) Note:

- $[B, A] = BA - AB = -(AB - BA) = -[A, B]$
- $[\sigma_i, \sigma_i] = \sigma_i^2 - \sigma_i^2 = 0$

Thus we only need to consider, $(i, j) \in \{(1,2), (1,3), (2,3)\}$

$$\begin{aligned} [\sigma_1, \sigma_2] &= \sigma_1\sigma_2 - \sigma_2\sigma_1 = \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \\ &= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} - \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix} = 2i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 2i\sigma_3 \end{aligned}$$

and

$$2i\sum_{k=1}^{3} \epsilon_{12k}\sigma_k = 2i\epsilon_{123}\sigma_3 = 2i\sigma_3$$

$$\begin{aligned} [\sigma_1, \sigma_3] &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = -2i\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = -2i\sigma_2 \end{aligned}$$

and

$$2i\sum_{k=1}^{3} \epsilon_{13k}\sigma_k = 2i(\epsilon_{131}\sigma_1 + \epsilon_{132}\sigma_2 + \epsilon_{133}\sigma_3) = -2i\sigma_2$$

$$\begin{aligned} [\sigma_2, \sigma_3] &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \\ &= \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} - \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} = 2i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2i\sigma_1 \end{aligned}$$

and

$$\sum_{k=1}^{3} \epsilon_{23k}\sigma_k = 2i\epsilon_{131}\sigma_1 = 2i\sigma_1$$

$$\begin{aligned} 2i\sigma_3 &= [\sigma_1,\sigma_2] = -[\sigma_2,\sigma_1] = -2i\sum_{k=1}^{3}\epsilon_{21k}\sigma_k = \\ &= -2i\epsilon_{213}\sigma_3 = 2i\sigma_3 \end{aligned}$$

c) Similarly

$$\begin{aligned} \{\sigma_i,\sigma_i\} &= \sigma_i^2+\sigma_i^2 = 2\sigma_i^2 = 2I = 2I\delta_{ii}, \qquad i=1,2,\dots \\ &\text{and} \\ \{\sigma_i,\sigma_j\} &= \{\sigma_j,\sigma_i\} \\ &\text{and} \\ \{\sigma_1,\sigma_2\} &= \sigma_1\sigma_2+\sigma_2\sigma_1 = 0 \\ \{\sigma_1,\sigma_3\} &= \sigma_1\sigma_3+\sigma_3\sigma_1 = 0 \\ \{\sigma_2,\sigma_3\} &= \sigma_2\sigma_3+\sigma_3\sigma_2 = 0 \end{aligned}$$

2. We begin by writing the 3 linear equations (3),(4),(5) as the augmented matrix,

$$\left[\begin{array}{ccc|c} 6 & 18 & 1 & 20 \\ -1 & -3 & 8 & 4 \\ 5 & 15 & -9 & 11 \end{array}\right] \begin{array}{c} R1\to R3 \\ R3\to R2 \\ R2\to R1 \end{array} \sim \left[\begin{array}{ccc|c} -1 & -3 & 8 & 4 \\ 5 & 15 & -9 & 11 \\ 6 & 18 & -4 & 20 \end{array}\right] \sim$$

$$\sim_{\substack{R2=5R1+R2\\R3=6R1+R3}} \left[\begin{array}{ccc|c} -1 & -3 & 8 & 4 \\ 0 & 0 & 40-9 & 11+20 \\ 0 & 0 & 44 & 44 \end{array}\right] \sim_{\substack{R3=R3/44\\R2=R2/44}}$$

$$\sim_{R1=-R1} \left[\begin{array}{ccc|c} 1 & 3 & -8 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \sim_{R3=R3+R2} \left[\begin{array}{ccc|c} 1 & 3 & 8 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim$$

$$\sim_{R1=R1+8R2} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Which corresponds to the row equivalent linear system,

$$x_1 + 3x_2 = 4$$

$$x_3 = 1$$

Letting $x_2 = t$ implies that the general solution set is given by,

$$(\star) = \begin{array}{l} x_1 = -3t+4 \\ x_2 = t \\ x_3 = 1 \end{array}, \qquad t \in \Re$$

With x_1 dependent on the <u>one</u> free variable x_2
($\star$) parameterizes a 2-D line in 3-D space

3.

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 3 & h & k \end{array}\right] \sim^{R3=R3-3R1} \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & h-9 & k-6 \end{array}\right]$$

corresponds to the linear system

$$x_1 + 3x_2 = 2$$
$$(h-9)x_2 = k-6$$

a) For this system to be consistent with a unique solution,

$$(h-9)x_2 = k-6 \Rightarrow x_2 = \frac{k-6}{h-9}, \text{ assuming } h-9 \neq 0$$

thus $h - 9 \neq 0 \Rightarrow h \neq 9$ will yield no free variables and the linear system is consistent with a unique point of intersection of the two lines.

b) For infinitely many solutions (for x_2 to be a free variable) we require that

$$(h-9)x_2 = k-6 \Leftrightarrow 0 \cdot x_2 = 0 \Rightarrow h = 9, k = 6$$

Thus x_2 is free.

c. For no solutions we require,

$$(h-9)x_2 = k-6 \Leftrightarrow 0 \cdot x_2 = c, c \in \Re, c \neq 0$$

This implies that $h = 9$ and $k \neq 6$. Thus the augmented column is a pivot column and the system has no solutions.

4. We have the following augmented matrix representation of (1) and (2),

$$\left[\begin{array}{cc|c} a & b & f \\ c & d & g \end{array}\right] \sim^{R1=dR1-bR2}_{R1=aR2-cR1} \left[\begin{array}{cc|c} ad-cb & bd-db & df-bg \\ ac-ca & ad-cb & dg-cf \end{array}\right] =$$

$$=^{(\star)} \left[\begin{array}{cc|c} ad-cb & 0 & df-bg \\ 0 & ad-cb & ag-cf \end{array}\right] \sim^{R2=R2/(ad-cb)}_{R1=R1/(ad-cb)} \left[\begin{array}{cc|c} 1 & 0 & \frac{df-bg}{ad-cb} \\ 0 & 1 & \frac{ag-cf}{ad-cb} \end{array}\right]$$

which is equivalent to the linear system,

$$(1') \qquad 1 \cdot x_1 + 0 \cdot x_2 = x_1 = fracdf - bgad - cb$$
$$(2') \qquad 0 \cdot x_1 + 1 \cdot x_2 = \frac{ag-cf}{ad-cb}$$

Note, to do the division at ($\star$) we have assumed that

$$ad - bc \neq 0$$

This is a common statement which places restriction on a,b,c,d.

5. If $\vec{b} = \begin{bmatrix} 22 \\ 20 \\ 15 \end{bmatrix}$ is a linear combination of the vectors, $\vec{a}_1 = \begin{bmatrix} 5 \\ -4 \\ 9 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} 3 \\ 7 \\ -2 \end{bmatrix}$ formed from the columns of $A = \begin{bmatrix} 5 & 3 \\ -4 & 7 \\ 9 & -2 \end{bmatrix}$ then there must exist $x_1, x_2 \in \Re$ such that

$$\vec{a}_1 \cdot x_1 + \vec{a}_2 \cdot x_2 = \vec{b} \Leftrightarrow \begin{array}{l} 5x_1 + 3x_2 = 22 \\ -4x_1 + 7x_2 = 20 \\ 9x_1 - 2x_2 = 15 \end{array}$$

To determine if this is true we row reduce the augmented matrix,

$$\left[\begin{array}{cc|c} 5 & 3 & 22 \\ -4 & 7 & 20 \\ 9 & -2 & 15 \end{array}\right] \sim_{\substack{R3=5R3-9R1 \\ R2=5R2+4R1}} \left[\begin{array}{cc|c} 5 & 3 & 22 \\ 0 & 47 & 188 \\ 0 & -37 & -123 \end{array}\right] \sim$$

$$\sim_{R3=47R3+37R2} \left[\begin{array}{cc|c} 5 & 3 & 22 \\ 0 & 47 & 188 \\ 0 & 0 & 1175 \end{array}\right]$$

which corresponds to the linear system,

$$\begin{array}{ll} 5x_1 + 3x_2 = 22 & \\ 47x_2 = 188 & \\ 0 \cdot x_2 = 1175 & (\star) \end{array}$$

There is no x_2 such that $(\star)$ can be satisfied. Thus, the linear system is inconsistent and $\vec{b}$ is <u>not</u> a linear combination of $\vec{a}_1$ and $\vec{a}_2$.