MATH348-Advanced Engineering Mathematics		**Homework: LA Part II**
	Introduction to Vector Spaces, Eigenproblems and Diagonalization	
Text: 7.4, 7.9,	Lecture Notes: N/A	Slides: N/A

Quote of Homework: Linear Algebra Part II
Don Juan Matus: The answer is very simple. He must not run away. He must defy his fear, and in spite of it he must take the next step in learning, and the next, and the next. He must be fully afraid, and yet he must not stop. That is the rule!
Carlos Castaneda - The Teachings of Don Juan: A Yaqui Way of Knowledge (1968)

1. Vocabulary of Vector Spaces

Given,

$$\mathbf{A}_1 = \begin{bmatrix} 5 & 3 \\ -4 & 7 \\ 9 & -2 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 22 \\ 20 \\ 15 \end{bmatrix},$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix},$$

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 5 \\ -7 \\ h \end{bmatrix},$$

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix},$$

$$\mathbf{A}_2 = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

1.1. **Linear Combinations.** Is $\mathbf{b}_1$ a linear combination of the columns of $\mathbf{A}_1$?

1.2. **Linear Dependence.** Determine all values for h such that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ forms a linearly dependent set.

1.3. **Linear Independence.** Determine all values for h such that $S = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ forms a linearly independent set.

1.4. **Spanning Sets.** How many vectors are in $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$? How many vectors are in span(S)? Is $\mathbf{y} \in$span(S)?

1.5. **Matrix Spaces.** Is $\mathbf{b}_2 \in \text{Nul}(\mathbf{A}_2)$? Is $\mathbf{b}_2 \in \text{Col}(\mathbf{A}_2)$?

2. The Spaces Defined by Linear Transformations

Given,

$$\mathbf{A} = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix}.$$

2.1. **Null Space.** Determine a basis and the dimension of $\text{Nul}(\mathbf{A})$.

2.2. **Column Space.** Determine a basis and the dimension of $\text{Col}(\mathbf{A})$.

2.3. **Row Space.** Determine a basis and the dimension of Row $\mathbf{A}$. What is the Rank of $\mathbf{A}$?

3. Eigenvalues and Eigenvectors

$$\mathbf{A}_1 = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}, \quad \mathbf{A}_4 = \begin{bmatrix} .1 & .6 \\ .9 & .4 \end{bmatrix}, \quad \mathbf{A}_5 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},$$

3.1. **Eigenproblems.** Find all eigenvalues and eigenvectors of $\mathbf{A}_i$ for $i = 1, 2, 3, 4, 5$.

3.2. **Diagonlization.** Find all matrices associated with the diagonalization of $\mathbf{A}_i$ for $i = 3, 4, 5$.

4. Regular Stochastic Matrices

For the *regular stochastic matrix* $\mathbf{A}_4$, define its associated steady-state vector, $\mathbf{q}$, to be such that $\mathbf{A}_4\mathbf{q} = \mathbf{q}$.

4.1. **Limits of Time Series.** Show that $\lim_{n\to\infty} \mathbf{A}_4^n\mathbf{x} = \mathbf{q}$ where $\mathbf{x} \in \mathbb{R}^2$ such that $x_1 + x_2 = 1$.

5. Orthogonal Diagonalization and Spectral Decomposition

Recall that if $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ then their inner-product is defined to be $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^{\mathrm{H}}\mathbf{y} = \bar{\mathbf{x}}^{\mathrm{T}}\mathbf{y}$. In this case, the 'length' of the vector is $|\mathbf{x}| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$.

5.1. **Self-Adjointness.** Show that $\mathbf{A}_5$ is a self-adjoint matrix.

5.2. **Orthogonal Eigenvectors.** Show that the eigenvectors of $\mathbf{A}_5$ are orthogonal with respect to the inner-product defined above.

5.3. **Orthonormal Eigenbasis.** Using the previous definition for length of a vector and the eigenvectors of the self-adjoint matrix, construct an *orthonormal basis* for $\mathbb{C}^2$.

5.4. **Orthogonal Diagonalization.** Show that $\mathbf{U}^{\mathrm{H}} = \mathbf{U}^{-1}$, where $\mathbf{U}$ is a matrix containing the normalized eigenvectors of $\mathbf{A}_5$.

5.5. **Spectral Decomposition.** Show that $\mathbf{A}_5 = \lambda_1\mathbf{x}_1\mathbf{x}_1^{\mathrm{H}} + \lambda_2\mathbf{x}_2\mathbf{x}_2^{\mathrm{H}}$.