E. Kreyszig, Advanced Engineering Mathematics, 9^{th} ed.	Section 7.8, pgs. 315-322
Lecture: Contruction and Properties of $\mathbf{A}^{-1}$	Module: 04
Suggested Problem Set: Suggested Problems : {7, 8, 15, 17, 19}	January 19, 2009

E. Kreyszig, Advanced Engineering Mathematics, 9^{th} ed.	Section 7.7, pgs. 308-315
Lecture: Determinants and Cramer's Rule	Module: 05
Suggested Problem Set: Suggested Problems : {5, 6, 13, 15, 16, 19, 24(a,b,c)}	January 19, 2009

Quote of Lecture 4

Smile and grin at the change all around me. Pick up my guitar and play. Just like yesterday.

The Who: Won't Get Fooled Again (1971)

Up to this point we have built some logic and intuition supporting the idea that to methodically solve a linear system efficiently we rewrite the system as a matrix vector product, $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}_{m\times n}$ is a matrix containing the coefficient data for the system and $\mathbf{b}$ is an inhomogeneity, which corresponds to translations of the linear equations in space. The goal now is to find the vector unknown $\mathbf{x}$ and to do this we apply the row-reduction algorithm to the corresponding augmented matrix $[\mathbf{A}|\mathbf{b}]$. [1]

We say that if $m < n$ then there are fewer equations than unknowns and that the system is *under-determined* and expect that at best there may be infinitely many solutions.[2] If $m > n$ then there are more equations than unknowns and the system is *over-determined* and in this case we expect that solutions may *exist* and possibly be *unique*.[3] The final case, $m = n$, provides more intuition about the behavior of solutions to linear systems. In the following we recap the results for square systems (as many equations as unknowns) and add to this growing list of equivalent statements:

1. There exists a unique solution, $\mathbf{x} \in \mathbb{R}^{n\times 1}$ to the system $\mathbf{Ax} = \mathbf{b}$ for every $\mathbf{b} \in \mathbb{R}^{n\times 1}$.
2. The homogeneous system $\mathbf{Ax} = \mathbf{0}$ has only the *trivial solution*, $\mathbf{x} = \mathbf{0}$.
3. $\mathbf{A} \sim \mathbf{I}$
4. There is a pivot in every column of the coefficient matrix.
5. The columns of $\mathbf{A}$ are **linearly independent**. [4]

So, from this we gather the following idea, 'If the solution to a square system, $\mathbf{Ax} = \mathbf{b}$, exists and is unique then there must exist an inverse matrix $\mathbf{A}^{-1}$ such that $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.' At this point our logic begs the questions:

- Given a square matrix $\mathbf{A}$, assuming that it's inverse, $\mathbf{A}^{-1}$, exists then how do we find it?[5]
- Suppose we don't want to actually find $\mathbf{A}^{-1}$. Is there a way to know that $\mathbf{A}^{-1}$ exists without finding it?[6]

[1] At this point it should be clear to the reader that if $\mathbf{A} \in \mathbb{R}^{m\times n}$ then $\mathbf{x} \in \mathbb{R}^{n\times 1}$ and $\mathbf{b} \in \mathbb{R}^{m\times 1}$.

[2] Quick sanity check: If two planes are placed in space then at best they will intersect at a line or be the same plane.

[3] See homework 1 problem 5 for a case where three-lines are given and that these three lines have a common intersection.

[4] We haven't yet used the term 'linearly independent', but we will.

[5] It turns out that one can find this matrix by the same process used to solve the linear system itself.

[6] There is a matrix function called the determinant. This function returns a scalar, which will tell us whether the columns of $\mathbf{A}$ are linearly independent and thus invertible.

Goals

- Understand the relationship between inverse matrices and solutions of linear systems.
- Connect the concept of determinants for square matrices with existence of inverses and solubility of square linear systems.

Objectives

- Define an algorithm for inverse matrices using it's proposed algebraic properties.
- Define and apply the cofactor expansion method for calculating determinants of square matrices.
- Connect the previous concepts by utilizing them to solve $\mathbf{Ax} = \mathbf{b}$.
- Record, without proof, some of the properties of inverse matrices and determinant calculations.