MATH348-Advanced Engineering Mathematics		**Homework: PDEs - Part I**
Known Solutions, Boundary Value Problems, The Heat Equation and PDE Readings		
Text: 12.1,12.3	Lecture Notes : 13-14	Lecture Slides: N/A

Quote of Homework Four
In life there an infinitely-many directions and each one is permitted.
CSM Professor Emeritus: John DeSanto - Mathematical Physics (2007)

1. Some Solutions to common PDE

Show that the following functions are solutions to their corresponding PDE's.

1.1. **Right and Left Travelling Wave Solutions.** $u(x,t) = f(x-ct) + g(x+ct)$ for the 1-D wave equation.

1.2. **Decaying Fourier Mode.** $u(x,t) = e^{-4\omega^2 t}\sin(\omega x)$ where $c = 2$ for the 1-D heat equation.

1.3. **Radius Reciprocation.** $u(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$ for the 3-D Laplace equation.

1.4. **Driving/Forcing Affects.** $u(x,y) = x^4 + y^4$ where $f(x,y) = 12(x^2+y^2)$ for the 2-D Poisson equation.

Note: The PDE in question are,

- Laplace's equation : $\triangle u = 0$
- Poisson's equation : $\triangle u = f(x,y,z)$
- Heat/Diffusion Equation : $u_t = c^2 \triangle u$
- Wave Equation : $u_{tt} = c^2 \triangle u$

and can be found on page 563 of Kryszig - 9^{th} Edition.

2. Our Fundamental Boundary Value Problem

Boundary value problems (BVP) typically arise within the context of PDE, which are equations modeling the evolution of a quantity in both space and time. There are important general results for BVP, which are set within the context of Sturm-Liouville problems. What can be efficiently done by hand tends to be limited. The problem, in Cartesian coordinates, is to find all solutions to,

$$y'' + \lambda y = 0,\ \lambda \in \mathbb{R},\ x \in (0,L), \tag{1}$$

which also satisfy,

$$l_1 y(0) + l_2 y'(0) = 0, \tag{2}$$

$$r_1 y(L) + r_2 y'(L) = 0. \tag{3}$$

This problem is intractable, by hand, for general values of l_1, l_2, r_1, r_2. However, the following set of values,

	l_1	l_2	r_1	r_2
Case I	1	0	1	0
Case II	0	1	0	1
Case III	1	0	0	1
Case IV	0	1	1	0

lead to BVP that can be solved by hand.

2.1. **BVP Hand Calculations.** Find all nontrivial eigenfunctions associated with the previous BVP for each of the four cases. [1]

[1] Case I and Case II will be discussed in class.

- Case III : From the previous table of functions, first show that $y(0) = 0$ implies that $c_1 = c_3 = c_5 = 0$. Next show that $y'(L) = 0$ implies that $c_4 = c_6 = 0$. This leaves just the sine function to deal with. Lastly, show that $y(x) = c_2 \sin(\sqrt{\lambda} x)$ satisfies the condition $y'(L) = 0$ for the specific values $\sqrt{\lambda} = (2n-1)\frac{\pi}{2L}$ where $n = 1, 2, 3, \ldots$.

3. Heat Equation on a closed and bounded spatial domain of $\mathbb{R}^{1+1}$

Consider the one-dimensional heat equation,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad , \tag{4}$$

$$x \in (0, L)\,, \qquad t \in (0, \infty)\,, \qquad c^2 = \frac{K}{\sigma\rho}. \tag{5}$$

Equations (4)-(5) model the time-evolution of the temperature, $u = u(x,t)$, of a heat conducting medium in one-dimension. The object, of length L, is assumed to have a homogenous thermal conductivity K, specific heat σ, and linear density ρ. That is, $K, \sigma, \rho \in \mathbb{R}^+$. If we consider an object of finite-length, positioned on say $(0, L)$, then we must also specify the boundary conditions[2],

$$u_x(0,t) = 0, u_x(L,t) = 0, . \tag{6}$$

Lastly, for the problem to admit a unique solution we must know the initial configuration of the temperature,

$$u(x, 0) = f(x). \tag{7}$$

3.1. **Separation of Variables : General Solution.** Assume that the solution to (4)-(5) is such that $u(x,t) = F(x)G(t)$ and use separation of variables to find the general solution to (4)-(5), which satisfies (6)-(7). [3]

3.2. **Qualitative Dynamics.** Describe how the long term behavior of the general solution to (4)-(7) changes as the thermal conductivity, K, is increased while all other parameters are held constant. Also, describe how the solution changes when the linear density, ρ, is increased while all other parameters are held constant.

3.3. **Fourier Series : Solution to the IVP.** Define,

$$f(x) = \begin{cases} \dfrac{2k}{L} x, & 0 < x \le \frac{L}{2}, \\ \dfrac{2k}{L}(L - x), & \frac{L}{2} < x < L \end{cases} \tag{8}$$

and for the following questions we consider the solution, u, to the heat equation given by, (4)-(5), which satisfies the initial condition given by (8). [4] For $L = 1$ and $k = 1$, find the particular solution to (4)-(5) with boundary conditions (6)-(7) for when the initial temperature profile of the medium is given by (8). Show that $\lim_{t\to\infty} u(x,t) = f_{avg} = 0.5$.[5]

4. Readings on the Heat/Diffusion Equation

Consider the following readings:[6]

- `http://en.wikipedia.org/wiki/Heat_equation`
- `http://en.wikipedia.org/wiki/Heat_equation#Applications`
- `http://en.wikipedia.org/wiki/Black%E2%80%93Scholes`
- `http://en.wikipedia.org/wiki/Thermal_diffusivity`
- `http://en.wikipedia.org/wiki/Fick%27s_law`

- Case IV : From the previous table of functions, first show that $y'(0) = 0$ implies that $c_2 = c_4 = c_6 = 0$. Next show that $y(L) = 0$ implies that $c_3 = c_5 = 0$. This leaves just the cosine function to deal with. Lastly, show that $y(x) = c_1 \cos(\sqrt{\lambda}x)$ satisfies the condition $y(L) = 0$ for the specific values $\sqrt{\lambda} = (2n-1)\frac{\pi}{2L}$ where $n = 1, 2, 3, \ldots$.

[2] Here the boundary conditions correspond to perfect insulation of both endpoints

[3] An insulated bar is discussed in examples 4 and 5 on page 557.

[4] When solving the following problems it would be a good idea to go back through your notes and the homework looking for similar calculations.

[5] It is interesting here to note that though the initial condition f doesn't appear to satisfy the boundary conditions its periodic Fourier extension does. That is, if you draw the even periodic extension of the initial condition then you would see that the slope is not well defined at the end points. Remembering that the Fourier series averages the right and left hand limits of the periodic extension of the function f at the endpoints shows that the boundary conditions are, in fact, satisfied, since the derivative of an average is the average of derivatives.

[6] As with most wikipedia you will have to skim through the articles to figure out what is important to you. My advice would be to read the questions before looking at the wikipedia articles. Also, I wouldn't worry about reading the first article past the introduction.

4.1. **Questions about the Readings.** Please respond to each of the following.

1. What is thermal diffusivity and what does it measure about a physical object? How might you measure the thermal diffusivity of an object?
2. Consider the Black-Scholes equation and the Diffusion Equation, what are their dependent variables (AKA unknown functions) and what are their independent variables. In a sentence or two, what do each model?
3. How does one get a nonlinear diffusion equation from a linear one? Specifically, what changes to the physical parameters must be made so that a nonlinear equation is the result? Explain this in terms of the heat equation.
5. What is Brownian motion and how is the heat equation related to it?
6. What, from the heat equation, must you transform to get Schrödinger's equation from quantum mechanics?
7. Describe Fick's first law in terms of heat flow.
8. Where does the diffusion coefficient come from in Fick's first law and what does this constant measure?
9. Wikipedia says that the diffusion equation is more general than the heat equation. Do you agree of disagree with this statement? Justify your choice.

5. Readings on the Wave Equation

Consider the following readings:

- http://en.wikipedia.org/wiki/Wave_equation
- http://en.wikipedia.org/wiki/Dispersion_%28optics%29
- http://en.wikipedia.org/wiki/Wave_equation#From_the_generic_scalar_transport_equation

5.1. **Questions about the Readings.** Please respond to each of the following.

1. What is the mathematical difference between the heat equation and wave equation?
2. What does the physical parameter in the wave equation measure?
3. Is the wave equation dispersive and what is dispersion? Provide a physical example.
4. What changes must be made to the have equation to introduce nonlinearity? What might the nonlinear equation model?
5. What is the generic scalar transport equation? Specifically, how does it relate to the derivation of the heat equation in class?
6. What is advection as compared to convection? How does this relate to the solution to the scalar wave equation?