# Complex representation of waves

E-field is a real quantity (can measure)

$$E(x,t) = A\cos(kx - \omega t - \theta)$$

or

$$= Re\left[ A e^{i(kx - \omega t - \theta)} \right]$$

$$= \tfrac{1}{2} A e^{+i(\;)} + \tfrac{1}{2} A e^{-i(\;)}$$

$$= \tfrac{1}{2} A e^{+i(\;)} + c.c.$$

Much easier to work w/ complex notation (phasors).
- take real pt. in end.

complex field:

write $E(x,t) = E_0 e^{i(kx-\omega t)}$

with $E_0 = \underset{\text{ampl}}{A}\, e^{-i\theta}$ ↳ phase.

summary:
- harmon. solutions are the building blocks Fourier series, transforms.
- phase is critical → interference.
- other combinations can be used (diff't basis) e.g. wavelets.

Waves in 3D

$$\frac{\partial^2 E}{\partial x^2} - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} = 0 \rightarrow \nabla^2 E - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} = 0$$

(still treat E as a scalar for now)

pick geometry (typically according to boundary conditions

Cartesian $\partial_x^2 E + \partial_y^2 E + \partial_z^2 E - \mu\varepsilon\, \partial_t^2 E = 0$

separable solution

$$E(\vec{r}, t) = A f(x) g(y) h(z) l(t)$$

$$\rightarrow \quad = A e^{i k_x x} e^{i k_y y} e^{i k_z z} e^{-i\omega t}$$

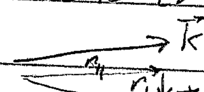
$$= A e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$\vec{k}$ gives direction of plane wave ("ray")

$\lambda = 2\pi / |k|$

Along planes $\perp \vec{k}$ phase is constant (wave front)

$\vec{k}\cdot\vec{r} = k_o r_{\parallel}$

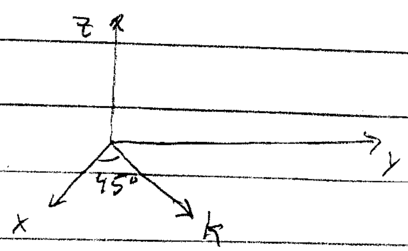



$\vec{r}_{\perp}$ can vary w/o change in $\vec{k}\cdot\vec{r}$

example:

plane wave traveling w/ $\vec{k}$ in x-y plane, 45° to x axis

$\therefore k_z = 0 \quad |\vec{k}| \equiv k_o \quad k_x = k_o/\sqrt{2} \quad k_y = k_o/\sqrt{2}$



Vector properties of EM plane waves

$\vec{k}$ defines direction of wave

suppose $\vec{k} = k_o \hat{z}$. For a plane wave, no variation in $x, y$

$$\vec{\nabla} \cdot \vec{E} = \cancelto{0}{\partial_x E_x} + \cancelto{0}{\partial_y E_y} + \partial_z E_z = 0$$

b/c no variation in $x, y$

$\partial_z E_z = 0$ means $E_z = 0$ $\vec{E} \perp \vec{k}$

const $E$ (DC) doesn't propagate.

similarly,

$$\nabla \cdot \vec{B} = 0 \rightarrow B_z = 0 \qquad \vec{B} \perp \vec{k}$$

if $\vec{E} = E_o \hat{y} e^{i(k_o z - \omega t)}$

use $\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$ to get $\vec{B}$

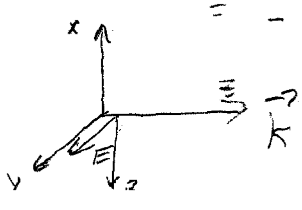
$$\nabla \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & E_y & 0 \end{vmatrix} = -\hat{x}\,\partial_z E_y + \hat{z}\,\cancelto{0}{\partial_x E_y}$$

$$\partial_t \vec{B} = -c\left(-i k_o E_o \hat{x} e^{i(k_o z - \omega t)}\right)$$

$$\vec{B} = i c k_o E_o \hat{x} e^{i k_o z} \int e^{-i\omega t}\, dt + \text{const}$$

$$= \frac{i c k_o}{-i\omega} E_o e^{i(k_o z - \omega t)} \hat{x}$$

$$= -E_o \hat{x} e^{i(k_o z - \omega t)} \qquad \omega = k_o c$$




$\vec{E} \perp \vec{B} \perp \vec{k}$

TEM wave with $B_o = E_o$

Note $E, B$ are in phase, same magnitude.

Normally represent with E field, since $\vec{E}$ has stronger effect on charges:

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$

since $|B| = |E|$

$$F \sim q\left(E + \frac{v}{c} E\right),$$ B interaction is less by $v/c$

Generalize for any $\vec{k}$:

rotate $\vec{k}$ from $\hat{z}$ to arbitrary direction.

$\vec{E}$ moves too:

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

each component of $\vec{E}$ has same functional form.

Now $$\nabla \cdot \vec{E} = \partial_x E_x + \partial_y E_y + \partial_z E_z$$
$$= i(k_x E_x + k_y E_y + k_z E_z) = 0$$

$$\rightarrow \boxed{\vec{k} \cdot \vec{E} = 0}$$

similarly

$$\nabla \times \vec{E} = i\vec{k} \times \vec{E} = -\frac{1}{c}(-i\omega)\vec{B}$$

$$\boxed{\vec{k} \times \vec{E} = \frac{\omega}{c}\vec{B}}$$

true for any plane wave in free space.

Wave intensity

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{H} \qquad \text{for } \mu = 1 \quad \vec{H} = \vec{B}$$

want to calculate this from $\vec{E}$ field:

$$\vec{k} \times \vec{E} = \frac{\omega}{c} \vec{B}$$

$$\vec{S} = \frac{c}{4\pi} \frac{c}{\omega} \vec{E} \times (\vec{k} \times \vec{E})$$

IP: $\vec{A} \times (\vec{B} \times \vec{C}) = B(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

$$\vec{E} \times (\vec{k} \times \vec{E}) = \vec{k}(\vec{E} \cdot \vec{E}) - \vec{E}(\underbrace{\cancel{\vec{E} \cdot \vec{k}}}_{0})$$

$$\therefore \vec{S} = \frac{c^2}{4\pi} \frac{\vec{k}}{\omega} \vec{E} \cdot \vec{E}$$

in a medium $\omega = k_o c = (k_o n)\left(\frac{c}{n}\right)$

$$\vec{S} = \frac{c^2}{4\pi} \frac{k_o n}{k_o c} \hat{k} (\vec{E} \cdot \vec{E})$$

$$\vec{S} = \hat{k} \frac{nc}{4\pi} (\vec{E} \cdot \vec{E}) \qquad \text{direction of } \hat{k}$$

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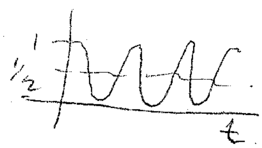
dot product : important to handle polarization states time average:

with real fields

$$\langle \vec{S} \rangle \propto \langle E_0^2 \cos^2 \omega t \rangle$$

$$\langle \cos^2 \omega t \rangle = \frac{1}{T}\int_0^T \cos^2 \omega t \, dt$$




$$= 1/2$$

either by setting $T = 2\pi/\omega$ or limit as $T \to \infty$

with complex fields

$$\langle \vec{S} \rangle = \frac{1}{2} \vec{E}^* \cdot \vec{E} \cdot \frac{nc}{4\pi} \hat{k}$$

this correctly eliminates $i\omega t$

see HM 5.3 for cautions about taking real part.

Alt expr:

$$\langle S \rangle = v_{ph} \langle \mathcal{E} \rangle \hat{k}$$

use caution when using these for

- non-monochromatic
- non-isotropic
- non linear