E. Kreyszig, Advanced Engineering Mathematics, 9^{th} ed.	Section 12.5, pgs. 552-562
Lecture: Heat Equation - Solution by Fourier Series	Module: 14
Suggested Problem Set: {13, 18}	November 30, 2009

Beginning Quote of Lecture 14	
The Mark Inside was coming up on him and that's a heat nobody can cool.	
	William S. Burroughs : Naked Lunch (1959)

1. Introduction

The box-header implies we seek to only study the solutions of the heat equation that can be written as a Fourier series. While this is true there is more that needs to be done. It makes sense to begin with a physically motivated starting location and build up the problem definition so that we know exactly what our equation models.[1] After this we consider how we might find solutions to the model equation and, more importantly, what these solutions *mean* for our model. If we have done everything right then we can port much of this discussion over to the study of vibrations and simple wave motion.

2. Conservation Laws

We begin the story with some sub-region Ω of $\mathbb{R}^3$, in which we are concerned about a quantity u that is allowed to move freely through this sub-region. We take this quantity to be the temperature at some space-time point (x, y, z, t) in Ω. [2] Now, within this region Ω we restrict our study to some ball $B \subset \Omega$ and within this ball we now argue that:

- The total rate of change of heat energy inside the ball must be equal to the heat energy produced inside the ball minus the heat energy that leaves the ball through its boundary.

This is a statement of conservation of energy and without this argument we wouldn't even have an equation to start with. In terms of mathematics the previous statement provides the following equation,

$$\frac{d}{dt}\iiint_B \rho\, c\, u(x,y,z,t)\, dV = \iiint_B f(x,y,z,t)\, dV - \iint_{\partial B} \vec{\phi}(x,y,z,t)\cdot \hat{n}\, dS, \rho, c \in \mathbb{R}^+ \tag{1}$$

where ρ, c are constants representing the density and heat-capacity of the homogeneous material and f is a generic function meant to represent the rate of heat generation at any of the points in the ball. The vector $\vec{\phi}$ is called the *heat-flux* vector and represents the direction and rate associated with the flow of heat energy in the ball. In order to compare the integrands in the previous equation we make use of the divergence theorem to convert the flux integral to a volume integral of the divergence of the vector field, $\text{div}(\vec{\phi}) = \vec{\nabla}\cdot\vec{\phi}$ and arrive at the *global conservation law*,

$$\iiint_B \left[\rho\, c\, u_t - f + \vec{\nabla}\cdot\vec{\phi}\right] dV = 0. \tag{2}$$

We now argue that since the ball B was an arbitrary geometry and that this integral must hold regardless of the geometry of B then it must hold point-wise and from this we conclude that the integrand itself must be zero, which gives us the *local conservation law*,

$$\rho\, c\, u_t - f + \vec{\nabla}\cdot\vec{\phi} = 0. \tag{3}$$

If we wish to work from either of these conservation laws we will immediately encounter a consistency problem. That is, we have only one equation, the conservation law, but we have two unknown variables u

[1]It turns out that from this perspective the model equation is quite generic and can be be used for a wide variety of physical phenomenon.

[2]Though we consider u to be a temperature it doesn't have to be. In fact we only require u to be a density of 'stuff.' This stuff could be charge, particle, probability or just about anything else we are permitted to construct an average of. In our case u is an average of the kinetic energy present at the some point $(x, y, z, t) \in \mathbb{R}^{3+1}$. For this reason the conservation law here is also found in the study of particle diffusion and electromagnetic theory.

and $\vec{\phi}$.[3] In order to deal with this we must appeal to a constitutive relation that relates the unknown density u to its flux vector $\vec{\phi}$. Specific to this problem we ask the question:

- How does heat energy flow with respect to the local temperature?

The answer to this question comes from our good friend Joseph Fourier but is not the only constitutive relation one can apply to our conservation law.

3. Constitutive Relations

A constitutive relation is an auxiliary equation used to close our system of equations, which at this time stands at one equation with two unknowns. These relations are derived, typically, from empirical evidence and vary depending on the physical context. [4] In our case we have what is known as Fourier's law of heat conduction:[5]

- The rate and direction of heat transfer is proportional the negative of the gradient of the temperature field.

Mathematically we take this to mean,

$$\vec{\phi} = -\kappa\vec{\nabla}u, \tag{4}$$

where $\kappa \in \mathbb{R}^+$ is a physical parameter associated with the thermal conductivity of the material. Using this relation in our previous conservation law we have the following partial differential equation,

$$\frac{\partial u}{\partial t} = \alpha^2 \triangle u + F(x, y, z, t), \tag{5}$$

where $\alpha^2 = \kappa/\rho c$, $F = f/\rho c$ and $\triangle u$ is known as the Laplacian of u and in Cartesian coordinates is $\triangle u = u_{xx} + u_{yy} + u_{zz}$. [6] This equation is called the heat or diffusion equation and models the flow of the conserved density (temperature) in space-time subject to the empirically observation that the density flows in down spatial gradients. This problem can take place in finite or infinite domains. Between these we will concentrate on finite domains and recall that to move to an infinite domain would imply a movement from Fourier series to Fourier transform/integral, which brings extra sophistication due to the more intensive integrations.

4. Half-Range Expansions and Finite-Domains

We wish to study the time-dynamics of temperature in an object, which is allowed to experience only lateral heat flow and no internal sources of heat energy. Mathematically, this will take place in one-spatial dimension and one-temporal dimension, which implies we use the model equation,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial t^2}, \quad c^2 \in \mathbb{R}^2. \tag{6}$$

This equation will control the time dynamics of the temperature of the object. If we take it to be L-units long then we say $x \in (0, L)$ and generally $t \in (0, \infty)$. However, this is not the entire story. We must specify how this object is to interact with the rest of its environment. These conditions are called boundary conditions and for this problem we specify,

$$u(0, t) = 0, \; u(L, t) = 0, \tag{7}$$

---

[3] The parameters ρ and c would be prescribed, or guessed on, at the start of the problem. The same is true for the source function f.

[4] A more systematic way of doing this is given to us by the Clausius-Duhem inequality of continuum mechanics, which is a re-expression of the second law of thermodynamics that allows one to derive constitutive relations among the processes important quantities.

[5] In the study of diffusion this is also known as Fick's first law and relates the diffusive flux to the concentration field, by postulating that the flux goes from regions of high concentration to regions of low concentration, with a magnitude that is proportional to the concentration gradient (spatial derivative).

[6] It is important to note that application of the multivariate chain rule is required to take this differential operator into other coordinate systems. Some important cases are:

- Polar :$u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} + u_{zz}$
- Spherical : $u_{rr} + 2r^{-1}u_{rr} + r^{-2}u_{\theta\theta} + (r^2\sin^2(\theta))^{-1}u_{\phi\phi} + \cot(\theta)r^{-1}u_\theta$

In each of these equations the process of solving the associated PDE requires the use of Power-Series.

and take this to mean that the temperature at both endpoints is fixed to be zero for all time.[7] This implies that the object itself is situated in some sort of heat bath that can take any amount of heat energy from these contact points instantly. [8] Lastly, we need to describe how the temperature is configured initially. Since, we have a first order in time problem we a required to specify only one initial condition,

$$u(x,0) = f(x). \tag{8}$$

These three statements constitute a well-posed problem and with them we can write down both a general solution to the problem (6)-(7) as well as a solution to this initial value problem (6)-(8).

We solve this problem using a three part process. This is outlined in class and homework solutions so I will only present an overview here and state what the general conclusions are. The process will require:

- Separation of Variables : Assume that $u(x,t) = F(x)G(t)$ and using this reduce (6) to two ODEs one on time and one on space.

$$G'(t) + \lambda c^2 G(t) = 0 \tag{9}$$
$$F''(x) + \lambda F(x) = 0 \tag{10}$$

  The spatial equation is known as a boundary value problem (BVP) and is new to most of us. There will be infinitely many solutions to this problem and these solutions are nothing more than modes of a Fourier series expansion of u.

- Solve ODEs : If we solve the time and space ODE's we find that,

$$G_n(t) = B_n e^{-\lambda_n c^2 t}, \text{ where } B_n \in \mathbb{R}, \lambda_n = \frac{n^2\pi^2}{L^2}, n = 1,2,3,\ldots, \tag{11}$$
$$F(x) = \sin(\sqrt{\lambda_n}x), \quad \sqrt{\lambda_n} = \frac{n\pi}{L}, n = 1,2,3,\ldots \tag{12}$$

- Form the solution to the IVP via superpostion to get,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(\sqrt{\lambda_n}x)e^{-\lambda_n c^2 t}, \tag{13}$$

  where $u(x,0) = f(x)$ implies that $B_n = \frac{2}{L}\int_0^L f(x)\sin(\sqrt{\lambda_n}x)dx$.

The take home message here is that though the BVP was weird it really just gave us the spatial terms in a Fourier series, which we could have guessed at from the start. Since we are talking about a function defined on a finite spatial domain it makes sense to assume that there is a Fourier series representation for this function. Moreover, since the left boundary condition requires the function pass through the origin we can expect a Fourier sine series for this function. That is we can guess that the solution should look something like,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) G_n(t), \tag{14}$$

where G_n is some function we suppose controls the time dynamics of each Fourier mode. If we substitute this back into the PDE then we arrive at,

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) G_n'(t) = -c^2 \sum_{n=1}^{\infty} B_n \left(\frac{n\pi}{L}\right)^2 \sin\left(\frac{n\pi}{L}x\right) G_n(t). \tag{15}$$

If we equate terms on the left and terms on the right then we can conclude G_n must satisfy the ODE,

$$G_n'(t) = -c^2\left(\frac{n\pi}{L}\right)^2 G_n(t), \tag{16}$$

and we can conclude the same solution as above and we didn't really have to deal with the BVP. All we have to use is the heady but fair assumption of periodic extension.

[7] If we instead specified the boundary conditions, $u_x(0,t) = u_x(L,t) = 0$, we would imply from Fourier's law of heat conduction that the heat-flux at the endpoints is zero for all time. This implies that no heat-energy can leave this object and it is perfectly insulated from its environment. Both of these cases are quite ideal and maybe we wish to specify that $u_x(0,t) \propto u(0,t)$. This mixed condition implies that the insulation is proportional to the temperature and though it is more realistic it leads to problems that cannot be solved by hand. :(

[8] Maybe we want to think that there are huge heat-sinks at both ends that just wick heat energy away from the object and are never outside of their operational parameters.

5. Lecture Goals

Our goals with this material will be:

- Understand what is meant by a conservation law and constitutive relation.
- Understand what the heat/diffusion equation models and how one can solve it using the periodic extension of a function defined on a finite domain. [9]

6. Lecture Objectives

The objectives of these lessons will be:

- Derive the heat equation from conservation of energy and Fourier's law of heat conduction.
- Construct solutions to the heat equation on a finite domain through the use of Fourier half-range expansions, which are found via separation of variables and ODE methods.

End Quote of Lecture 14	
Hustlers of the world, there is one Mark you cannot beat - the Mark Inside.	
	William S. Burroughs : Naked Lunch (1959)

[9] This really is the deepest concept in this course. If we consider the BVP then we find ourselves solving an eigenvalue/eigenvector problem, which has an infinite amount of each. Then since the problem is linear an arbitrary linear combination of solutions is also a solution and we have exploited everything from ODE's and linear algebra to solve the problem. The solution itself is a Fourier series, which we could have guess straight out of the gates using some geometry and the concept of periodic extension, and thus we can solve for all of the unknown coefficients in terms of Fourier coefficients!