MATH 225 - Differential Equations June 9 , 2008
Homework 8, Field 2008 **Due Date**: June 12, 2008

LAPLACE TRANSFORMS - SECOND-ORDER LINEAR EQUATIONS (REVISITED) - FORCED MASS-SPRING SYSTEMS

1. Calculate the Laplace transform of $f_1(t) = \sinh(at)$ and $f_2(t) = \cosh(at), a \in \mathbb{R}$.
   Hint: You may want to refer to the previous homework assignment for the definitions of $\sinh(x)$ and $\cosh(x)$ or you might find it more efficient to note that $\sin(ix) = \sinh(x)$ and $\cos(ix) = \cosh(x)$ and repeat the calculations we did in class to find $\mathfrak{L}\{\cos(kt)\}$ and $\mathfrak{L}\{\sin(kt)\}$, which will find the transform of f_1 and f_2 simultaneously as both the real and imaginary parts of the transform integral. Doing it this way should make the standard form for these transforms, `http://en.wikipedia.org/wiki/Laplace_transform#Table_of_selected_Laplace_transforms`, make more sense.

2. Using the definition of transform show the following relationships:
   (a) $\mathfrak{L}\{e^{at}f(t)\} = F(s-a)$
   (b) $\mathfrak{L}\{f(t-a)u_a(t)\} = e^{-as}F(s), \quad a \geq 0$
   (c) $\mathfrak{L}\{f(t)u_a(t)\} = e^{-as}\mathfrak{L}\{f(t+a)\}, \quad a \geq 0.$

3. Consider the following second-order linear ordinary differential equation with constant coefficients,
$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = \delta(t), \quad y(0) = 0, \; y'(0) = 0. \tag{1}$$
   Solve the IVP (1) for the following cases:
   (a) $a = 1$, $b = -2$, $c = -3$
   (b) $a = 1$, $b = 4$, $c = 4$
   (c) $a = 1$, $b = -4$, $c = 13$
   (d) $a = 1$, $b = 0$, $c = 9$

4. Given the following forced simple harmonic oscillator.
$$2\frac{d^2y}{dt^2} + 8y = 6\cos(\omega t), \quad y(0) = 1, \; y'(0) = -1. \tag{2}$$
   (a) Set $\omega = 1$ and find the solution to the initial value problem.
   (b) Set $\omega = 2$ and find the solution to the initial value problem.
   (c) Describe the differences in the long term behavior of the steady-state solution for each oscillator

5. Again we investigate the forced mass spring system given by,
$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = f(t), \quad m, b, k \in R^+ \cup \{0\}. \tag{3}$$
   (a) Suppose we have that $b = 0$, $y(0) = \alpha$, $y'(0) = \beta$ and $f(t) = A\delta_T(t)$, show that the solution to (3) subject to these constraints is given by,
$$y(t) = \alpha\cos(\omega t) + \frac{\beta}{\omega}\sin(\omega t) + \frac{A}{m\omega}u_T(t)\sin(\omega(t-T)), \tag{4}$$
   where $\omega^2 = \dfrac{k}{m}$.
   (b) Suppose that we wish to hit the mass in such a way that after the impact the oscillations stop. Show that for this to occur we must choose,
$$A = \frac{\alpha m\omega}{\sin(\omega T)} \tag{5}$$
$$T = \frac{1}{\omega}\arctan\left(-\frac{\alpha\omega}{\beta}\right). \tag{6}$$