Causality, impulse response + the Kramers-Kronig relations.

In calculating ε: $\vec{D} = \varepsilon\vec{E}$, we assume a monochromatic wave at ω. What we have is actually in the frequency domain

$$\vec{D}(\vec{r},\omega) = \varepsilon(\omega)\vec{E}(\vec{r},\omega)$$

$\varepsilon(\omega)$ represents the material response.

Transform to the time domain:

$$\vec{E}(\vec{r},\omega) = \int_{-\infty}^{\infty} \vec{E}(\vec{r},t)\, e^{i\omega t}\, dt$$

$$\vec{E}(\vec{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(\vec{r},\omega)\, e^{-i\omega t}\, d\omega$$

note sign, 2π conventions.

$$\vec{D}(\vec{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varepsilon(\omega)\, E(\vec{r},\omega)\, e^{-i\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \varepsilon(\omega)\, e^{-i\omega t}\int_{-\infty}^{\infty} E(\vec{r},t')\, e^{i\omega t'}\, dt'$$

note t' vs. t.

interchange $\int$ order:

$$\vec{D}(\vec{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt'\, \vec{E}(\vec{r},t')\int_{-\infty}^{\infty} d\omega\, \varepsilon(\omega)\, e^{-i\omega(t-t')}$$

let $\bar{\varepsilon}(t) = \mathcal{F}^{-1}\{\varepsilon(\omega)\}$

then

$$\vec{D}(\vec{r},t) = \int dt'\, \vec{E}(\vec{r},t')\, \bar{\varepsilon}(t-t') \qquad \text{convolution}$$

the field in the medium ($\vec{D}$) is the convolution of the input field with the impulse response $\bar{\varepsilon}(t)$

We just derived the convolution theorem:

$$\mathfrak{F}\{E(t) \otimes \bar{\varepsilon}(t)\} = E(\omega)\,\varepsilon(\omega)$$

It is more physically relevant to represent the response in terms of the atomic response:

let $G(\tau) = \mathfrak{F}^{-1}\{\varepsilon(\omega) - 1\} = \mathfrak{F}^{-1}\{4\pi \chi_e(\omega)\}$

Now $\vec{D}(\vec{r},t) = \vec{E}(\vec{r},t) + \int_{-\infty}^{\infty} G(t-t')\, \vec{E}(\vec{r},t')\, dt'$

If the system is causal, we'll have $G(\tau) = 0$ for $\tau < 0$

at $t=0$,

$$\vec{D}(\vec{r},0) = \vec{E}(\vec{r},0) + \int G(-t')\, E(\vec{r},t')\, dt'$$

$$= \vec{E}(\ ) + \int_{-\infty}^{0} G(-t')\, E(\vec{r},t')\, dt' \qquad \text{if } G(\tau)=0 \text{ for } \tau<0$$

that is, response depends on past only

classical electron model:

$$\varepsilon(\omega) - 1 = \frac{4\pi N_a e^2/m}{\omega_0^2 - \omega^2 - i\beta\omega}$$

Calculate impulse response:

$$G(\tau) = \mathcal{F}^{-1}\{\varepsilon(\omega) - 1\} = \frac{4\pi N_a e^2}{m}\frac{1}{2\pi}\int_{-\infty}^{\infty}(\omega_0^2 - \omega^2 - i\beta\omega)^{-1} \times e^{-i\omega\tau}\,d\omega$$

Must use contour integration: find poles.

$$\omega^2 + i\beta\omega - \omega_0^2 = 0$$

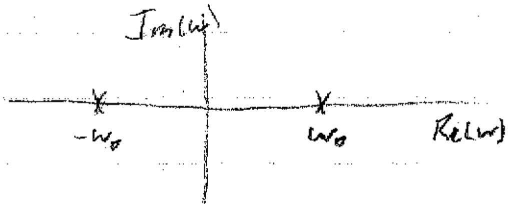
roots:

$$\omega_\pm = -\frac{i\beta}{2} \pm \frac{1}{2}\sqrt{-\beta^2 + 4\omega_0^2} = -\frac{i\beta}{2} \pm \omega_0'$$

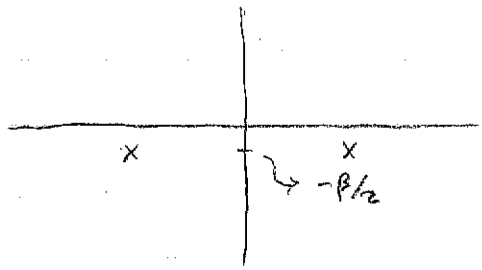
where $\omega_0' = \sqrt{\omega_0^2 - (\beta/2)^2}$

As $\beta \to 0$ poles are at $\omega = \pm\omega_0$

$\beta > 0$ pushes poles off real axis



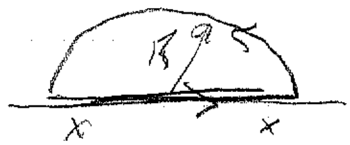
$\beta = 0$



$\beta > 0$

integrate along Re(ω) axis

diff't paths for $\tau < 0$, $\tau > 0$



$\tau < 0$: include path w/ $\mathrm{Im}(\omega) > 0$

since $e^{+i\omega|\tau|} \to 0$ for $R \to \infty$

i.e. $$G(\tau)\Big|_{\tau>0} = G(\tau) + \lim_{\omega\to\infty}\int_0^{\pi} \frac{e^{+i\omega|\tau|e^{-i\theta}}}{(\quad)}\,d\theta$$

0

No poles inside contour, so

$G(\tau) = 0$ for $\tau < 0$