Maxwell's Equations to the wave equation

- The induced polarization, **P**, contains the effect of the medium:

$$\vec{\nabla} \cdot \mathbf{E} = 0 \qquad \vec{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\vec{\nabla} \cdot \mathbf{B} = 0 \qquad \vec{\nabla} \times \mathbf{B} = \mu_0 \frac{\partial \mathbf{D}}{\partial t}$$

Define the displacement vector

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

- Derive the wave equation:

$$\rightarrow \vec{\nabla} \times \mathbf{B} = \mu_0 \frac{\partial \mathbf{D}}{\partial t} = \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \frac{\partial \mathbf{P}}{\partial t}$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \mathbf{E} \right) = -\vec{\nabla} \times \frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t} \left(\vec{\nabla} \times \mathbf{B} \right)$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \mathbf{E} \right) = \vec{\nabla} \left(\vec{\nabla} \bullet \mathbf{E} \right) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E}$$

Using: $\vec{\nabla} \cdot \mathbf{E} = 0$

Maxwell's Equations to the wave equation

- Finish derivation of the wave equation

$$\vec{\nabla} \times \left(\vec{\nabla} \times \mathbf{E} \right) = -\nabla^2 \mathbf{E} = -\frac{\partial}{\partial t} \left(\vec{\nabla} \times \mathbf{B} \right)$$

$$-\frac{\partial}{\partial t} \left(\vec{\nabla} \times \mathbf{B} \right) = -\frac{\partial}{\partial t} \left(\frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \frac{\partial \mathbf{P}}{\partial t} \right) = -\left(\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2} \right)$$

$$\boxed{\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2}}$$

"Inhomogeneous Wave Equation"

- For a plane wave traveling in the z-direction,

$$\frac{\partial^2 \mathbf{E}}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2}$$

Other geometries dictate how to deal with the Laplacian operator

Maxwell's Equations in a Medium

- The induced polarization, **P**, contains the effect of the medium:

$$\vec{\nabla}^2\mathbf{E}-\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}=\mu_0\frac{\partial^2\mathbf{P}}{\partial t^2}$$

- Sinusoidal waves of all frequencies are solutions to the wave equation
- For linear response, **P** will oscillate at the same frequency as the input.

$$\mathbf{P}(t)=\varepsilon_0\chi\mathbf{E}(t) \qquad \rightarrow\vec{\nabla}^2\mathbf{E}-\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}=\mu_0\varepsilon_0\chi\frac{\partial^2\mathbf{E}}{\partial t^2}$$

- Then once we know the susceptibility χ, we can calculate the dielectric constant and the refractive index:

$$\rightarrow\vec{\nabla}^2\mathbf{E}-\frac{1}{c^2}(1+\chi)\frac{\partial^2\mathbf{E}}{\partial t^2}=0 \qquad \varepsilon_0(1+\chi)=\varepsilon_0\varepsilon_r=\varepsilon_0 n^2$$

$$\vec{\nabla}^2\mathbf{E}-\frac{n^2}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}=0$$

Linear WE, anisotropic medium

- If the medium is *anisotropic*, i.e. birefringent, the magnitude of the induced polarization is still proportional to the incident field
  - But now the susceptibility is a tensor

$$\mathbf{P}\left(\mathbf{E}\right)=\varepsilon_0\overset{\leftrightarrow}{\chi}\cdot\mathbf{E},\quad \mathbf{D}=\varepsilon_0\mathbf{E}+\mathbf{P}=\varepsilon_0\left(1+\overset{\leftrightarrow}{\chi}\right)\cdot\mathbf{E}=\varepsilon_0\overset{\leftrightarrow}{\varepsilon}\cdot\mathbf{E}$$

  - In this case, the medium re-orients the direction of the displacement vector
  - If the coordinate system is chosen to diagonalize the dielectric tensor,

$$\overset{\leftrightarrow}{\varepsilon}=\begin{pmatrix}\varepsilon_{xx} & 0 & 0\\ 0 & \varepsilon_{yy} & 0\\ 0 & 0 & \varepsilon_{zz}\end{pmatrix}\qquad \overset{\leftrightarrow}{\chi}=\begin{pmatrix}\chi_{xx} & 0 & 0\\ 0 & \chi_{yy} & 0\\ 0 & 0 & \chi_{zz}\end{pmatrix}$$

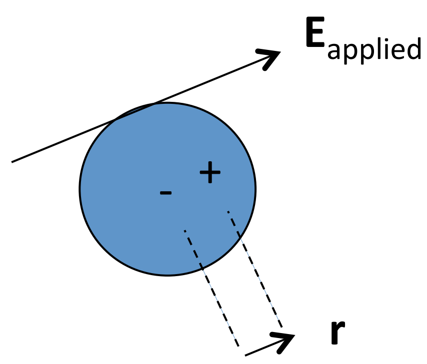
Connecting the macroscopic to the microscopic response

So determining the gain or loss coefficient depends on calculating the macroscopic induced polarization **P** or equivalently the susceptibility χ.

$$\mathbf{P}(\mathbf{E}) = \varepsilon_0 \chi \mathbf{E} = N_a \mathbf{p}$$

Note that the *macrosopic* polarization is really a density of individual dipole moments on the *microscopic* scale.

Recall: $\mathbf{p} = q\,\mathbf{r}$



So if the electric field is linearly polarized in the x-direction, then

$$\mathbf{P}(t) = N_a \mathbf{p}(t) = -N_a e x(t)$$

Here we treat x(t) as the position of the *electron*.

Spring model for dipole response

- Model: driven SHO with damping

$$m_e\ddot{x}(t) = -eE(t) - m_e\omega_0^2 x(t) - 2m_e\gamma\dot{x}(t)$$

$$m_e\ddot{x}(t) + m_e\gamma\dot{x}(t) + m_e\omega_0^2 x(t) = -eE_0 e^{-i\omega t}$$

Radiation damping term γ

Restoring force, resonant at ω_0

External driving field, specific ω

let $x(t) = x_0 e^{-i\omega t}$ *x* must oscillate at driving frequency, not at resonance frequency

$$-m_e\omega^2 x_0 - i\omega m_e\gamma x_0 + m_e\omega_0^2 x_0 = -eE_0$$

$e^{-i\omega t}$ drops out

$$x_0(\omega) = -\frac{e}{m_e}E_0\frac{1}{\omega_0^2 - i\omega\gamma - \omega^2} \equiv -\frac{eE_0}{m_e}\frac{1}{D(\omega)}$$

x is fcn of *t*, but the amplitude x_0 depends on ω

Spring model for dispersion

- Now we can go from the microscopic response $x(t)$ to the macroscopic χ and n

$$P(t) = -N_a e\,x(t) = \varepsilon_0 \chi^{(1)} E(t)$$

$$\rightarrow \chi^{(1)} = -\frac{N_a e\,x(t)}{\varepsilon_0 E(t)} = -\frac{N_a e\,x_0(\omega) e^{-i\omega t}}{\varepsilon_0 E_0 e^{-i\omega t}}$$

$$x_0(\omega) = -\frac{e}{m_e} E_0 \frac{1}{\omega_0^2 - i\omega\gamma - \omega^2} \equiv -\frac{e E_0}{m_e} \frac{1}{D(\omega)}$$

$$\chi^{(1)}(\omega) = -\frac{N_a e}{\varepsilon_0} \left(-\frac{e}{m_e} \frac{E_0}{D(\omega)} \right) \frac{1}{E_0} = \frac{N_a e^2}{\varepsilon_0 m_e} \frac{1}{D(\omega)}$$

Note that this gives us the *frequency response* of the system.

Spring model: refractive index

- Linear susceptibility yields the refractive index:

$$n^2(\omega) = 1 + \chi^{(1)}(\omega) = 1 + \frac{N_a e^2}{\varepsilon_0 m_e} \frac{1}{D(\omega)}$$

$$n^2(\omega) = 1 + \frac{N_a e^2}{\varepsilon_0 m_e} \frac{1}{\omega_0^2 - i\omega\gamma - \omega^2}$$

Refractive index is a *complex* quantity.

- Solve for the Re and Im parts:

$$n^2(\omega) = 1 + \frac{N_a e^2}{\varepsilon_0 m_e} \frac{\omega_0^2 - \omega^2 + i\omega\gamma}{\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\gamma^2} = 1 + \frac{N_a e^2}{\varepsilon_0 m_e} \frac{(\omega_0 + \omega)(\omega_0 - \omega) + i\omega\gamma}{(\omega_0 + \omega)^2(\omega_0 - \omega)^2 + \omega^2\gamma^2}$$

- Near a resonance, $\omega_0 + \omega \approx 2\omega_0$ $\quad \omega\gamma \approx \omega_0\gamma$

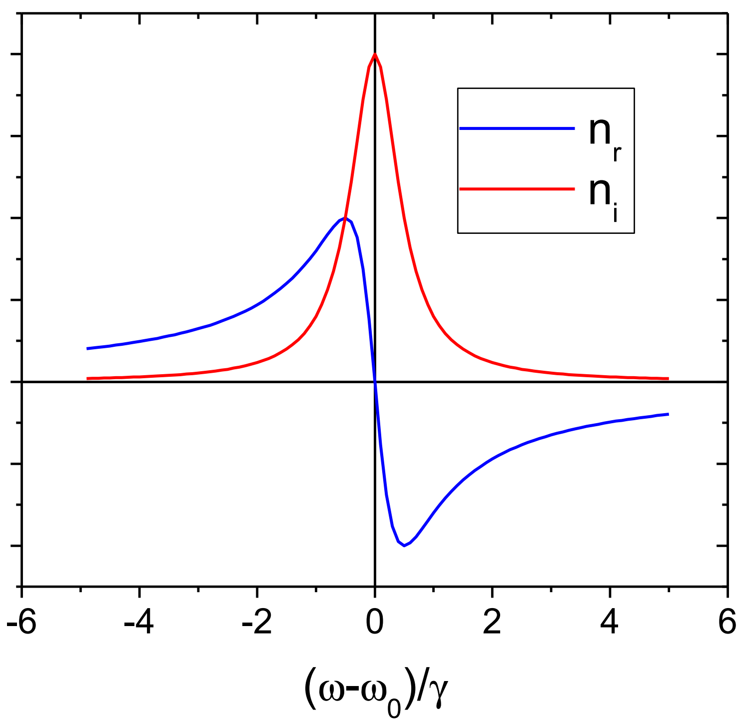
$$n^2(\omega) \approx 1 + \frac{N_a e^2}{\varepsilon_0 m_e} \frac{2\omega_0(\omega_0 - \omega) + i\omega_0\gamma}{4\omega_0^2(\omega_0 - \omega)^2 + \omega_0^2\gamma^2} = 1 + \frac{N_a e^2}{2\omega_0\varepsilon_0 m_e} \frac{(\omega_0 - \omega) + i\gamma/2}{(\omega_0 - \omega)^2 + (\gamma/2)^2}$$

Complex refractive index near resonance

$$n(\omega) \approx \sqrt{1 + \frac{N_a e^2}{2\omega_0 \varepsilon_0 m_e} \frac{(\omega_0 - \omega) + i\gamma/2}{(\omega_0 - \omega)^2 + (\gamma/2)^2}}$$

- For low atomic density (e.g. gas) $n \approx 1$

Normalized plot of n-1 and k versus $\omega - \omega_0$



$$n(\omega) \approx 1 + \frac{N_a e^2}{4\omega_0 \varepsilon_0 m_e} \frac{(\omega_0 - \omega) + i\gamma/2}{(\omega_0 - \omega)^2 + (\gamma/2)^2}$$

$$n_I(\omega) \approx \frac{N_a e^2}{4\omega_0 \varepsilon_0 m_e} \frac{\gamma/2}{(\omega_0 - \omega)^2 + (\gamma/2)^2}$$

$$n_R(\omega) \approx 1 + \frac{N_a e^2}{4\omega_0 \varepsilon_0 m_e} \frac{\omega_0 - \omega}{(\omega_0 - \omega)^2 + (\gamma/2)^2}$$

Lorentzian: FWHM = γ

Complex refractive index

- When the incident light is near resonance, both Re and Im parts of $n(\omega)$ are important.
  - What is the meaning of a complex refractive index?
  - For a plane wave propagating in the z-direction:

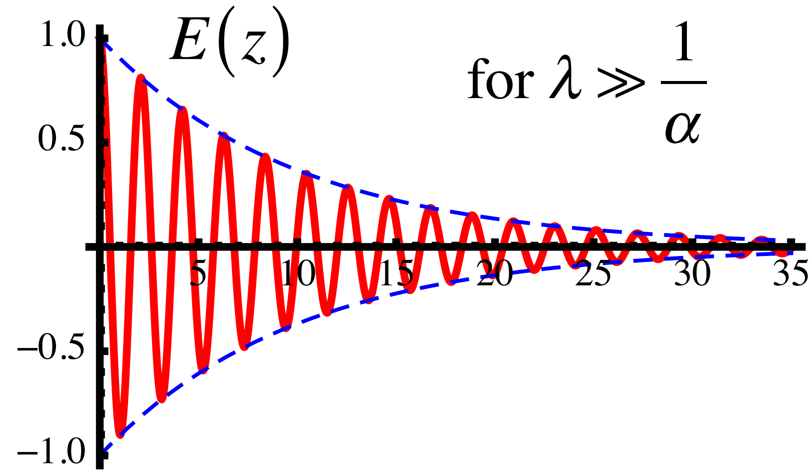
$$E(z,t) = E_0 e^{i(kz-\omega t)} = E_0 e^{i\left(\frac{\omega}{c}(n_R + i n_I)z - \omega t\right)} = E_0 e^{i\left(\frac{\omega}{c} n_R z - \omega t\right)} e^{-\frac{\omega}{c} n_I z}$$

For $n_I > 0$, absorption coefficient is

$$\alpha = \frac{\omega n_I}{2c}$$

For $n_I < 0$, gain coefficient is

$$g = \frac{\omega |n_I|}{2c}$$



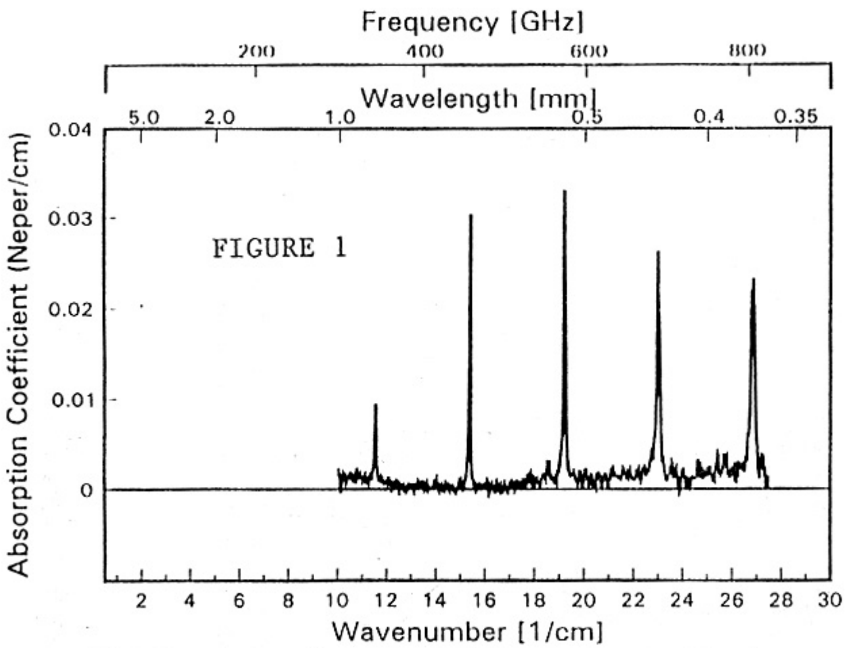
refractive index for real gases

- In a real atom or molecule, there are many resonances

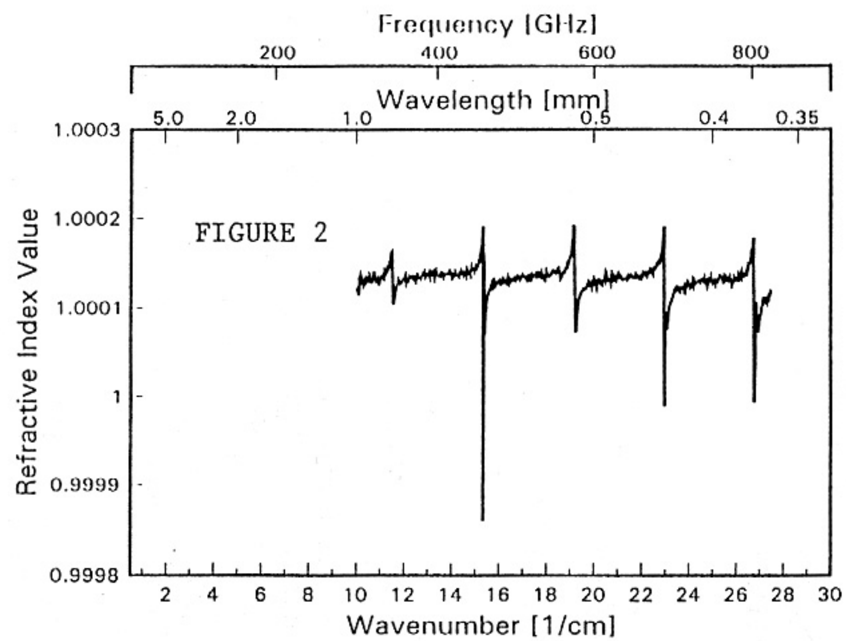
$$n^2 = 1 + \frac{N_a e^2}{\varepsilon_0 m_e} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2 - i\omega\gamma_j)}$$

f = oscillator strength

$$\sum_j f_j = Z$$



High Resolution Spectrum of Absorption Coeff. of Carbon Monoxide for 957mm path length(250 torr)



High Resolution spectrum of Refractive Index of Carbon Monoxide for 957mm path length(250 torr)

http://www.ece.tufts.edu/research/mm-smm/facility/gases.html

Transfer function and impulse response

- Treat the material as a linear system

$$E(z,\omega) = E_0(\omega)e^{ik(\omega)z} = E_0 e^{i\frac{\omega}{c}(n_R(\omega)+i n_I(\omega))z} = E_0 e^{i\frac{\omega}{c}n_R(\omega)z} e^{-\alpha(\omega)z}$$

- We could treat the whole expression in blue as the frequency response. But then it would be nonlinear with respect to z.
- For a physical interpretation, it is better to look at the impulse response of the microscopic system:

$$\mathbf{P}(\mathbf{E}) = \varepsilon_0 \chi(\omega)\mathbf{E}(\omega) = N_a \mathbf{p}(\omega)$$

$$\chi(\omega) = \frac{N_a e^2}{\varepsilon_0 m_e} \frac{1}{\omega_0^2 - i\omega\gamma - \omega^2}$$

Complex Lorentzian

Tilted window: ray propagation

- Calculate phase shift caused by the insertion of the window into an interferometer.
- Ray optics:
  - Add up optical path for each segment
  - Subtract optical path w/o window

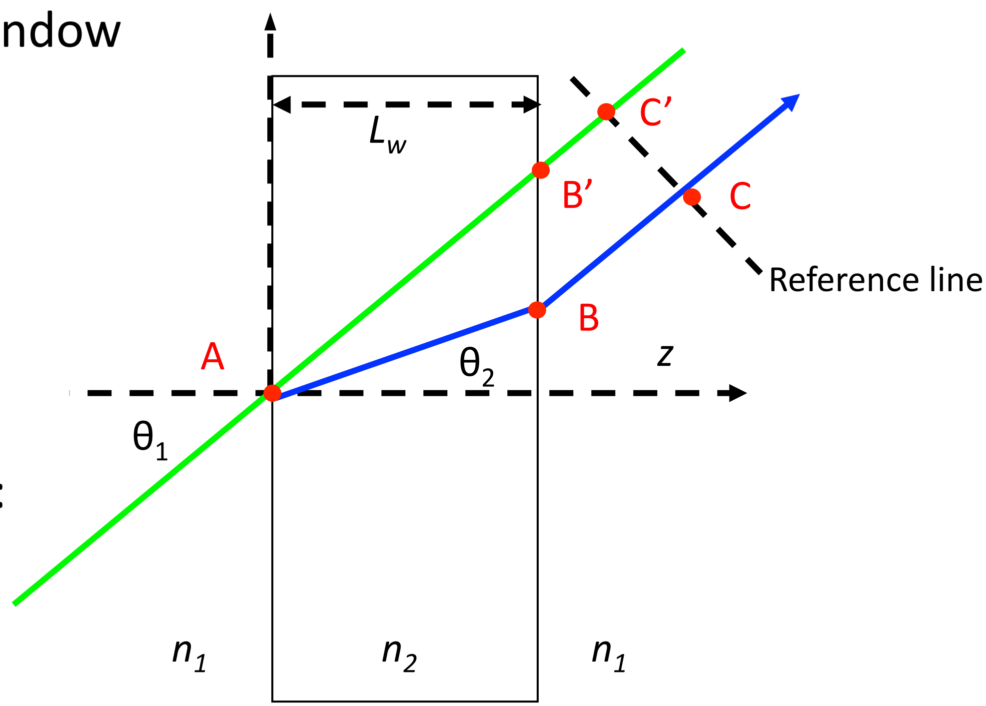
$$\Delta d = nL_{AB} + L_{BC} - L_{AB'} - L_{B'C'}$$

$$L_{AB} = \frac{L_w}{\cos\theta_2} \qquad L_{AB'} = \frac{L_w}{\cos\theta_1}$$

$$L_{BC} = L_{B'C'} + L_{BB'}\sin\theta_1$$

  - Use Snell's Law to reduce to:

$$\Delta d = n\,L_w\cos\theta_2 - L_w\cos\theta_1$$



Tilted window: wave propagation

- Write expression for tilted plane wave

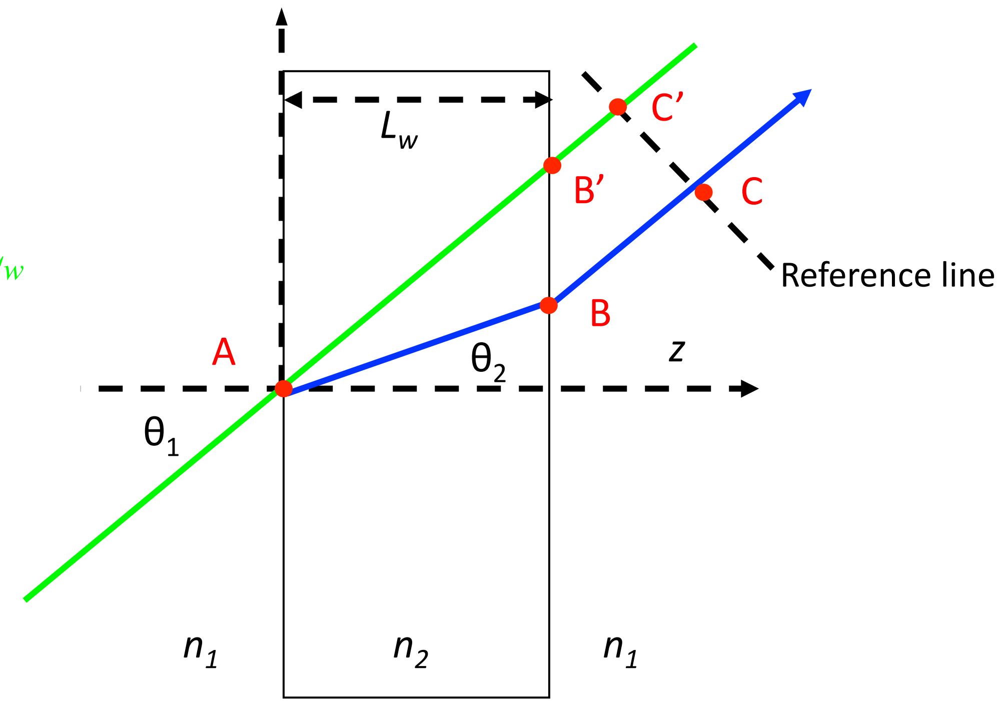
$$E(x,z)=E_0\exp\left[i\left(k_x x+k_z z\right)\right]=E_0\exp\left[i\frac{\omega}{c}n\left(x\sin\theta_2+z\cos\theta_2\right)\right]$$

- Snell's Law: phase across surfaces is conserved

$$k_x x=\frac{\omega}{c}n\sin\theta \qquad \textbf{is constant}$$

$$\Delta\phi=\left(k_2\cos\theta_2\right)L_w-\left(k_1\cos\theta_1\right)L_w$$

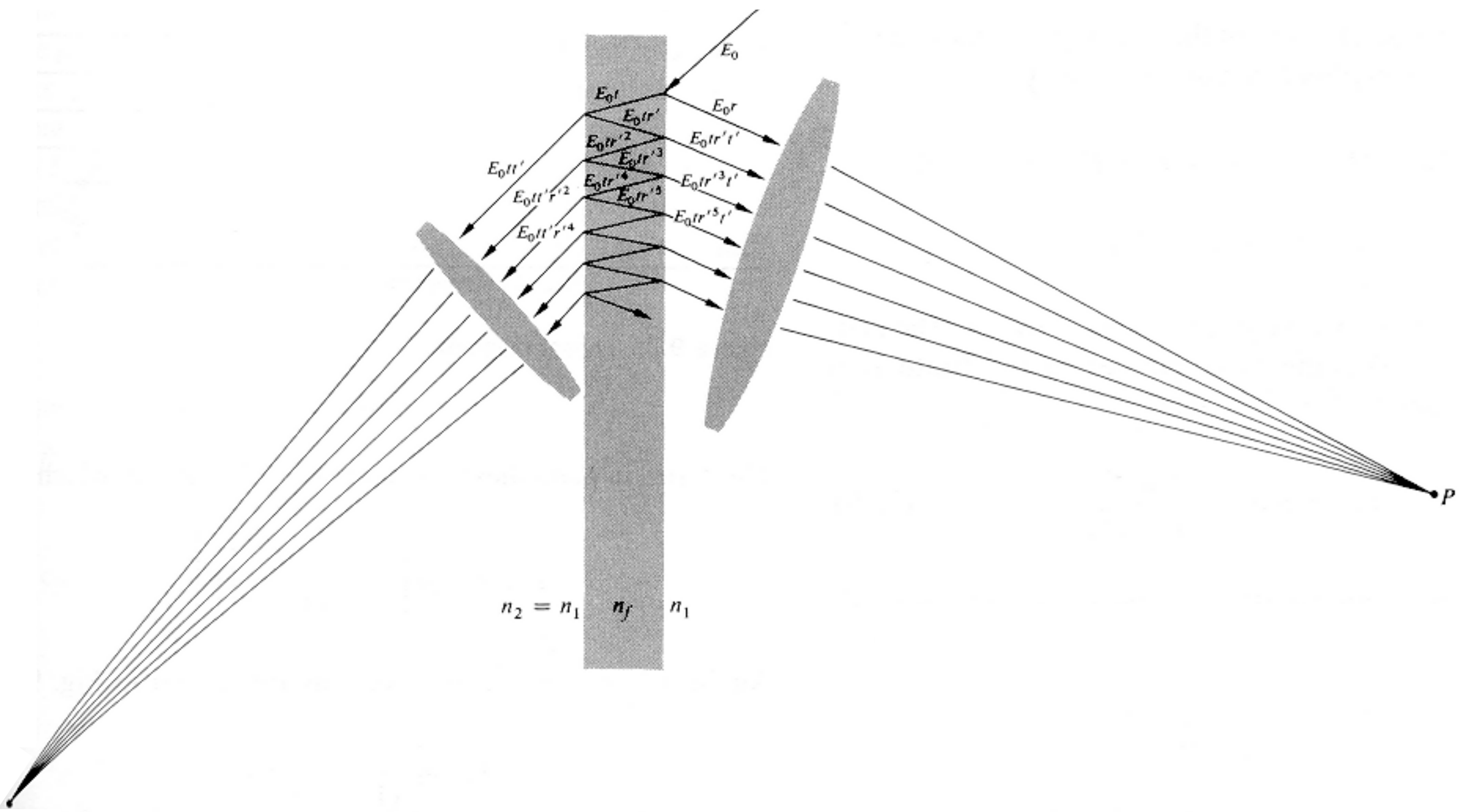
- This approach can be used to calculate phase of prism pairs and grating pairs



Multiple-beam interference:
The Fabry-Perot Interferometer or Etalon

A Fabry-Perot interferometer is a pair of **parallel** surfaces that reflect beams back and forth.
An etalon is a type of Fabry-Perot etalon, and is a piece of glass with parallel sides.

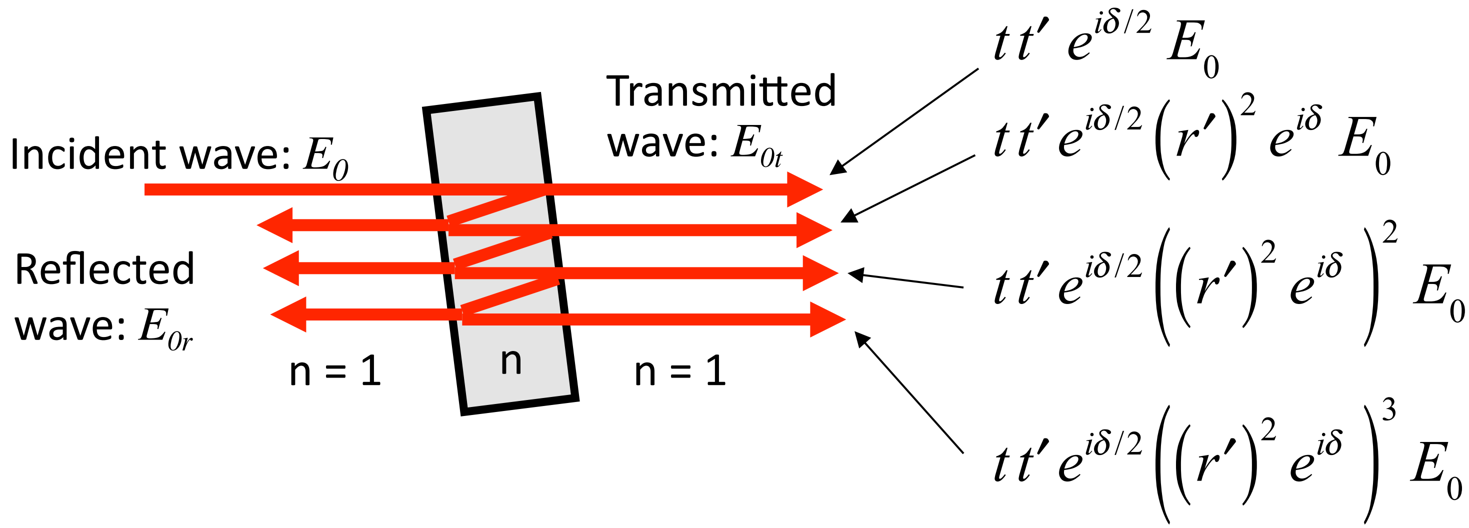
The transmitted wave is an infinite series of multiply reflected beams.



Linear systems approach to the FP

- As with any linear device, we can represent its action in either the temporal or frequency domain
  - Frequency domain: $H(\omega)$ = transfer function
  - Time domain: $h(t)$ = impulse response
- First we will start with the conventional approach:
  - Incident monochromatic plane wave
  - Calculate transmitted amplitude and phase for $H(\omega)$
- Then we should be able to calculate impulse response:
  - $h(t) = FT^{-1}\{ H(\omega) \}$

Calculation of the FP frequency response



r, t **= reflection, transmission coefficients from air to glass**
r', t' **= " " " from glass to air**

δ = round-trip phase delay inside medium = $k_0(2\ n\ L \cos\theta_t)$

$$\delta(\omega) = \omega\frac{2nL}{c}\cos\theta_t(\omega) \approx \omega\frac{2L}{c}\cos\theta_i \qquad \text{for n=1}$$

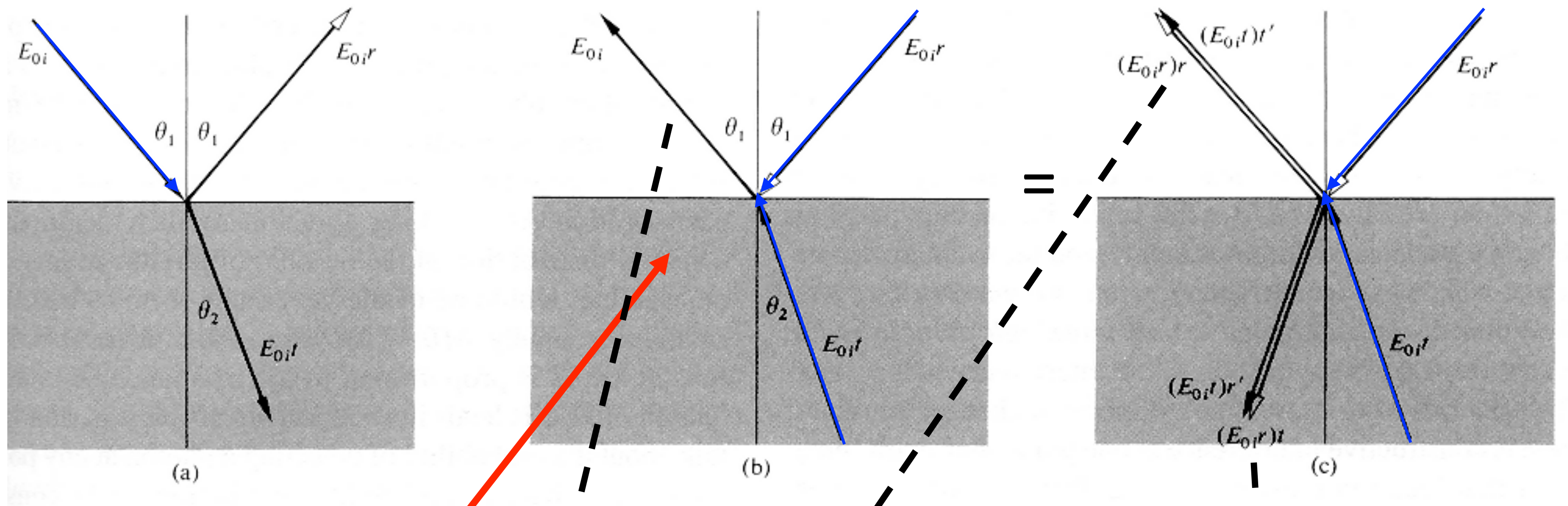
Transmitted wave:

$$E_{0t} = tt'e^{i\delta/2}E_0\left(1 + \left(r'\right)^2 e^{i\delta} + \left(\left(r'\right)^2 e^{i\delta}\right)^2 + \left(\left(r'\right)^2 e^{i\delta}\right)^3 + \ldots\right)$$

Reflected wave:

$$E_{0r} = rE_0 + tt'r'e^{i\delta}E_0 + tt'r'\left(\left(r'\right)^2 e^{i\delta}\right)^2 E_0 + \ldots$$

Stokes relations for reflection and transmission



"Time reversal:"
Same amplitudes,
reversed propagation
direction

$$E_{oi} = (E_{oi} r) r + (E_{oi} t) t'$$
$$\therefore tt' = 1 - r^2$$

$$(E_{oi} t) r' + (E_{oi} r) t = 0$$
$$\therefore r' = -r$$

Notes:

- **relations apply to angles connected by Snell's Law**
- **true for any polarization, but not TIR**
- **convention for which interface experiences a sign change can vary**

Fabry-Perot transfer function

Stokes' relations

$$r' = -r$$
$$r'^2 = r^2$$
$$tt' = 1 - r^2$$

The transmitted wave field is:

$$E_{0t} = tt' e^{i\delta/2} E_0 \left(1 + (r')^2 e^{i\delta} + \left((r')^2 e^{i\delta}\right)^2 + \left((r')^2 e^{i\delta}\right)^3 + \ldots\right)$$

$$= tt' e^{i\delta/2} E_0 \left(1 + r^2 e^{i\delta} + \left(r^2 e^{i\delta}\right)^2 + \left(r^2 e^{i\delta}\right)^3 + \ldots\right)$$

Perform sum over infinite series:

Let $x = r^2 e^{i\delta}$

$$E_{0t} = tt' e^{i\delta/2} E_0 \sum_{m=0}^{\infty} \left(r^2 e^{i\delta}\right)^m = \left(1 - r^2\right) e^{i\delta/2} E_0 \sum_{m=0}^{\infty} x^m$$

For |x|<1:□ $\sum_{m=0}^{\infty} x^m = 1 + x + x^2 + x^3 + \cdots = (1-x)^{-1}$

$$\Rightarrow E_{0t} = E_0 \frac{\left(1 - r^2\right) e^{i\delta/2}}{1 - r^2 e^{i\delta}} = E_0(\omega) \frac{\left(1 - r^2\right) \exp\left[i\omega \frac{L}{c} \cos\theta_i\right]}{1 - r^2 \exp\left[i\omega \frac{2L}{c} \cos\theta_i\right]} \equiv E_0(\omega) H(\omega)$$

FP impulse response

- Simple case: n = 1, normal incidence

$$H(\omega)=\frac{\left(1-r^2\right)\exp\left[i\omega\frac{L}{c}\cos\theta_i\right]}{1-r^2\exp\left[i\omega\frac{2L}{c}\cos\theta_i\right]}\rightarrow\frac{1-r^2}{1-r^2\,\mathrm{e}^{i\omega T_{RT}}}$$

Dropping common term:

$$e^{i\omega T_{RT}/2}$$

$$T_{RT}=2L/c$$

- FT^{-1} to get impulse response

$$H(\omega)=\frac{1-r^2}{1-r^2\,\mathrm{e}^{i\omega T_{RT}}}=\left(1-r^2\right)\sum_{m=0}^{\infty}\left(r^2e^{i\delta}\right)^m$$

$$h(t)=FT^{-1}\left\{H(\omega)\right\}=\left(1-r^2\right)\sum_{m=0}^{\infty}r^{2m}FT^{-1}\left\{e^{im\omega T_{RT}}\right\}$$

$$h(t)=\left(1-r^2\right)\sum_{m=0}^{\infty}r^{2m}\delta\left(t-mT_{RT}\right)$$

FP impulse response in high reflectivity limit

- When the reflectivity is high, very little is transmitted through the output on each reflection

$r^2 = R$ Power reflectivity

$R^m = (1-T)^m \approx (1-mT) \approx e^{-2m\gamma}$

Logarithmic cavity loss (single pass)

$\gamma = -\ln T = -\ln(1-R)$

- We can represent this as a time dependent loss function:

$L(t) = e^{-t/\tau_c}$ Cavity lifetime: $\tau_c = T_{RT} / 2\gamma$

$$h(t) = (1-r^2)\sum_{m=0}^{\infty} r^{2m}\delta(t - mT_{RT}) \rightarrow (1-r^2)\Theta(t)e^{-t/\tau_c}\,\mathrm{comb}(t/T_{RT})$$

where $\mathrm{comb}(t/T) \equiv \sum_{n=-\infty}^{\infty} \delta(t - n\,T)$

Fabry-Perot power transmission

$$E_{0t} = E_0 \frac{1-r^2}{1-r^2 e^{i\delta}} e^{i\delta/2}$$

Power transmittance:

$$T \equiv \left|\frac{E_{0t}}{E_0}\right|^2 = \left|\frac{\left(1-r^2\right)e^{i\delta/2}}{1-r^2 e^{i\delta}}\right|^2 = \frac{\left(1-r^2\right)^2}{(1-r^2 e^{+i\delta})(1-r^2 e^{-i\delta})}$$

$$= \left[\frac{(1-r^2)^2}{\{1+r^4-2r^2\cos(\delta)\}}\right] = \left[\frac{(1-r^2)^2}{\{1+r^4-2r^2[1-2\sin^2(\delta/2)]\}}\right] = \left[\frac{(1-r^2)^2}{\{1-2r^2+r^4+4r^2\sin^2(\delta/2)]\}}\right]$$

Dividing numerator and denominator by $(1-r^2)^2$

$$T = \frac{1}{1+F\sin^2\left(\delta/2\right)}$$

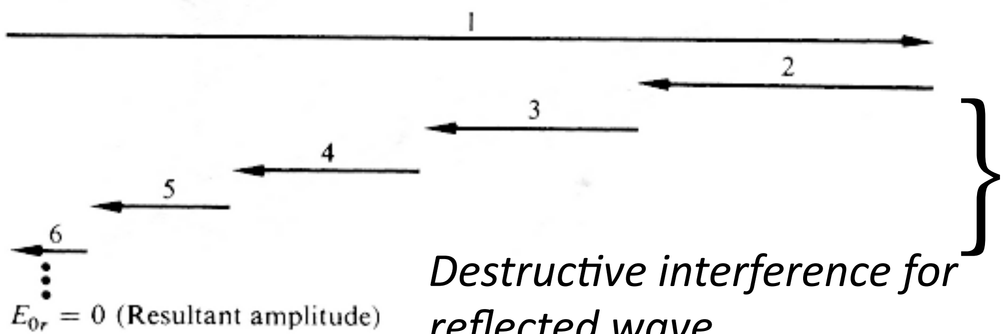
where: $F = \left[\frac{2r}{1-r^2}\right]^2$

Multiple-beam interference: simple limits

<u>Reflected waves</u>

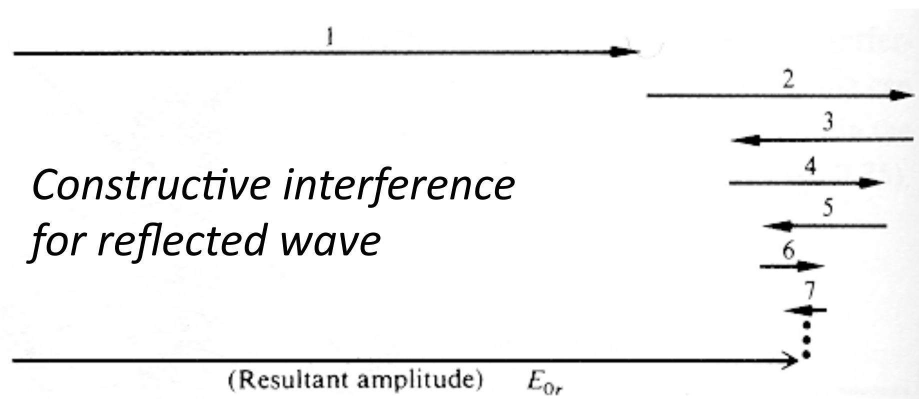
$$T = \frac{1}{1 + F\sin^2\left(\delta/2\right)}$$

Full transmission: sin( ) = 0, d = 2 π m



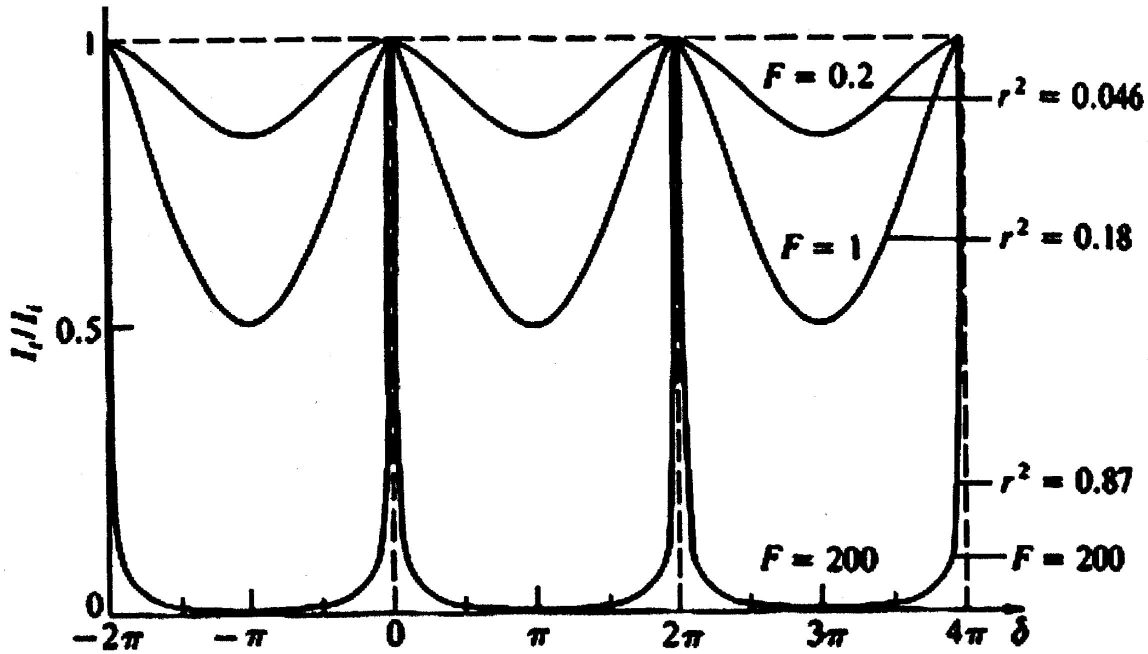
Destructive interference for reflected wave

Minimum transmission: sin( ) = 1, d = 2 π (m+1/2)



Constructive interference for reflected wave

Etalon transmittance vs. thickness, wavelength, or angle



$$T = \frac{1}{1 + F\sin^2(\delta/2)}$$

Transmission max: sin( ) = 0, d = 2 π m

$$\delta = \frac{\omega}{c} 2nL\cos[\theta_t] = 2\pi m$$

At normal incidence:

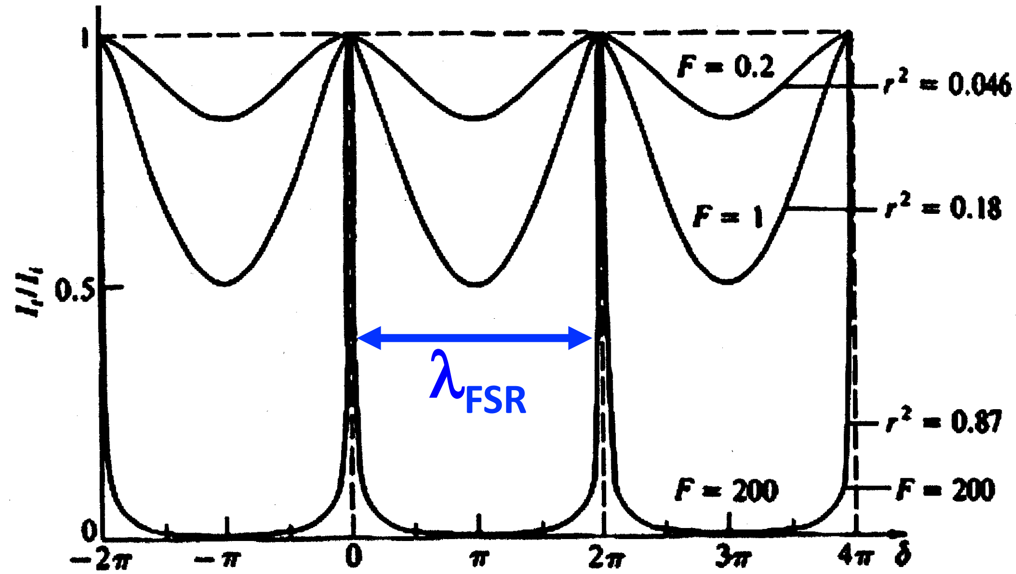
$$\lambda_m = \frac{2nL}{m} \quad \text{or} \quad nL = m\frac{\lambda_m}{2}$$

- The transmittance varies significantly with thickness or wavelength.
- We can also vary the incidence angle, which also affects δ.
- As the reflectance of each surface ($R=r^2$) approaches 1, the widths of the high-transmission regions become very narrow.

The Etalon Free Spectral Range

The Free Spectral Range is the wavelength range between transmission maxima.

λ_{FSR} = Free Spectral Range



For neighboring orders:

$$\frac{4\pi nL}{\lambda_1} - \frac{4\pi nL}{\lambda_2} = 2\pi \Rightarrow \frac{1}{\lambda_1} - \frac{1}{\lambda_2} = \frac{1}{2nL} = \frac{\lambda_2 - \lambda_1}{\lambda_1 \lambda_2}$$

$$\lambda_2 - \lambda_1 = \lambda_{FSR}$$

$$\lambda_2 \lambda_1 \approx \lambda^2$$

$$\lambda_{FSR} \approx \frac{\lambda^2}{2nL}$$

$$\frac{\lambda_{FSR}}{\lambda} = \frac{\lambda}{2nL} = \frac{\nu_{FSR}}{\nu}$$

$$\nu_{FSR} \approx \frac{c}{2nL}$$

1/(round trip time)

Etalon Linewidth

The Linewidth δ_{LW} is a transmittance peak's full-width-half-max (FWHM).

$$T = \frac{1}{1 + F\sin^2\left(\delta/2\right)}$$

A maximum is where $\delta/2 \approx m\pi + \delta'/2$ and $\sin^2\left(\delta/2\right) \approx \delta'/2$

Under these conditions (near resonance),

$$T = \frac{1}{1 + F\delta'^2/4}$$

This is a Lorentzian profile, with FWHM at:

$$\frac{F}{4}\left(\frac{\delta_{LW}}{2}\right)^2 = 1 \quad \Rightarrow \quad \delta_{LW} \approx 4/\sqrt{F}$$

This transmission linewidth corresponds to the minimum resolvable wavelength.

Etalon Finesse

The Finesse, $\Im$, is the ratio of the
Free Spectral Range and the Linewidth:

δ = 2 π corresponds to one FSR

$$\Im \equiv \frac{\delta_{FSR}}{\delta_{FW}} = \frac{2\pi}{4/\sqrt{F}} = \frac{\pi\sqrt{F}}{2}$$

Using: $$F = \left[\frac{2r}{1-r^2}\right]^2$$

$$\Im = \frac{\pi}{1-r^2}$$

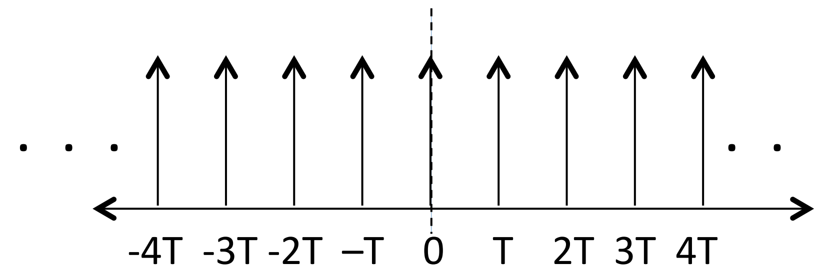
taking $r \approx 1$

The Finesse is the number of wavelengths the interferometer can resolve.

Comb function

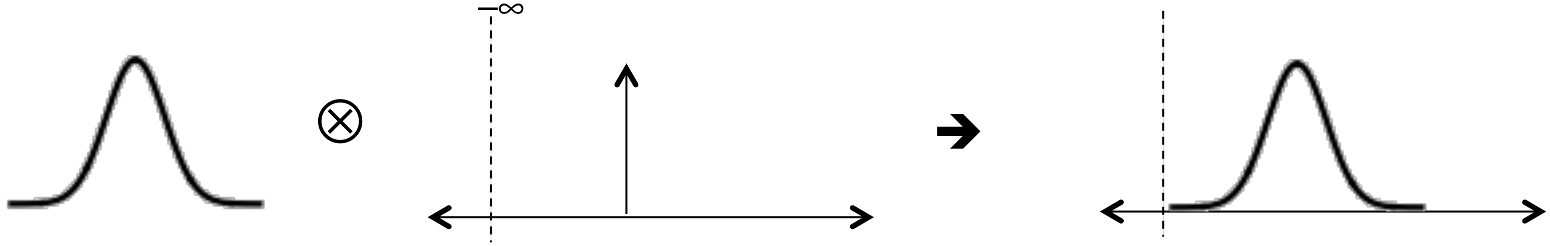
- Define the comb function

$$\mathrm{comb}\left(t/T\right) \equiv \sum_{n=-\infty}^{\infty} \delta\left(t - nT\right)$$

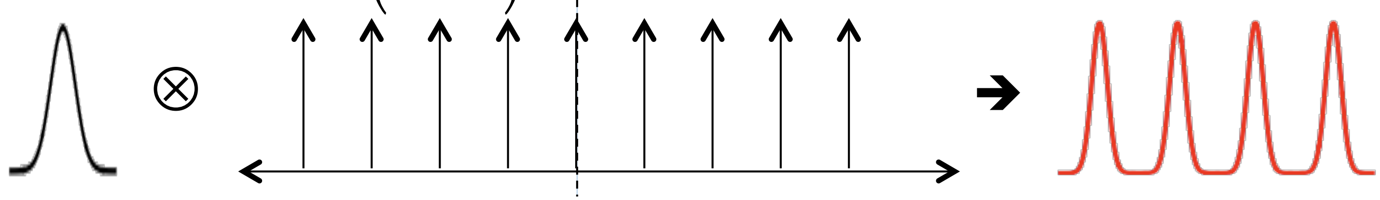


- A pulse train can be written as a convolution

$$f\left(t\right) \otimes \delta\left(t-T\right) = \int_{-\infty}^{\infty} f\left(t-t'\right)\delta\left(t'-T\right)dt' = f\left(t-T\right)$$



$$e^{-t^2/t_0^{\,2}} \otimes \mathrm{comb}\left(t/T\right)$$



Array theorem: FT of comb( )

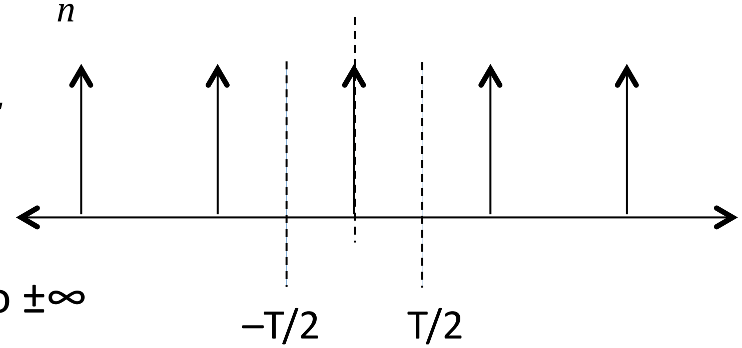
- Basic FT is straightforward:

$$f(t) = \text{comb}(t/T) \equiv \sum_{n=-\infty}^{\infty} \delta(t - nT) \qquad F(\omega) = \sum_{n=-\infty}^{\infty} FT\{\delta(t-nT)\} = \sum_{n=-\infty}^{\infty} e^{i\omega nT}$$

- This is actually a comb function also
- Since comb( ) is a periodic function (period T), we can write as a Fourier series:

$$f(t) = \sum_n c_n e^{i2\pi nt/T}$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-i2\pi nt/T} = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-i2\pi nt/T}$$



Integrate over one period, but we can extend integral to ±∞

$$c_n = \frac{1}{T} \qquad \therefore \text{comb}(t/T) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{2\pi nt/T}$$

Array theorem (cont)

- Now take FT:

$$f(t) = \text{comb}(t/T) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{2\pi nt/T}$$

$$F(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} FT\left\{e^{2\pi nt/T}\right\} = \frac{1}{T}\sum_{n=-\infty}^{\infty} 2\pi\delta\left(\omega + \frac{2\pi n}{T}\right)$$

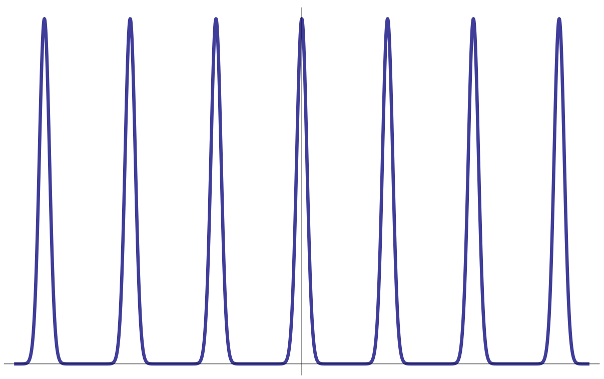
$$F(\omega) = \frac{2\pi}{T} comb\left(\frac{\omega}{2\pi/T}\right)$$

- So FT{comb} = comb
  - Frequency spacing $\Delta\omega=2\pi/T$ or $\Delta\nu=1/T$

Spectrum of a pulse train

- Gain envelope on longitudinal mode spectrum

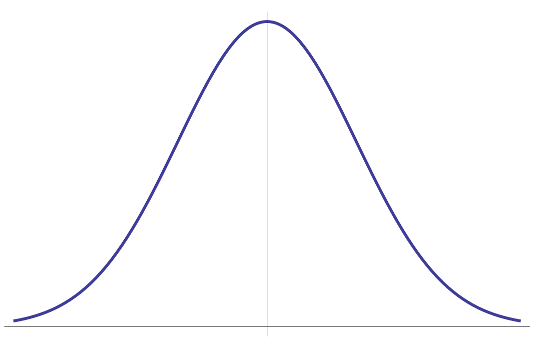
$$e^{-t^2/t_0^{\ 2}} \otimes \text{comb}\left(t/T\right)$$



$$FT\left\{e^{-t^2/t_0^{\ 2}} \otimes \text{comb}\left(t/T\right)\right\}$$

$$= e^{-t_0^{\ 2}\omega^2/4}\,\text{comb}\left(\omega/\Delta\omega\right)$$

T is pulse spacing = round trip time in laser resonator
Δv=1/T = spacing of peaks in frequency
= longitudinal mode spectrum



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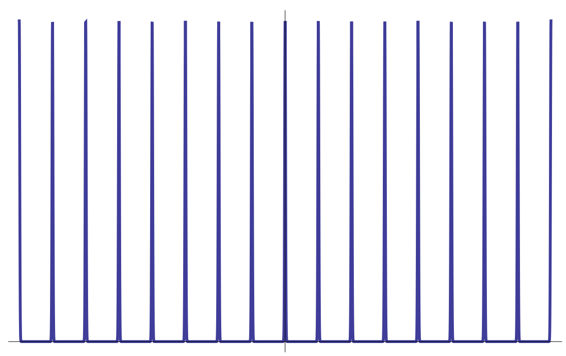

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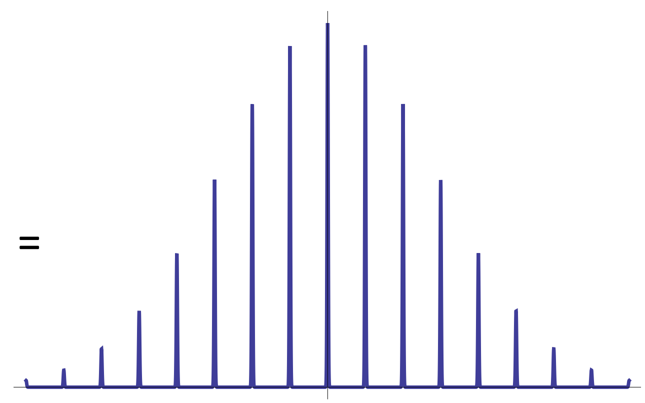


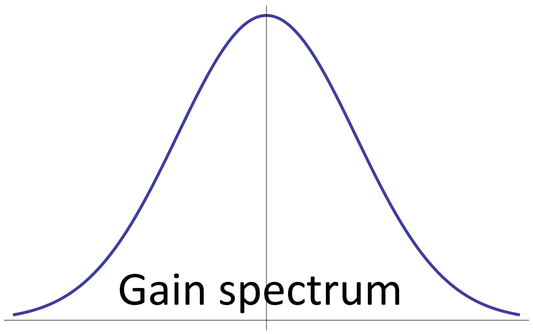
Spectrum of a pulse train

- Reverse reasoning: multiply gain envelope on longitudinal mode spectrum

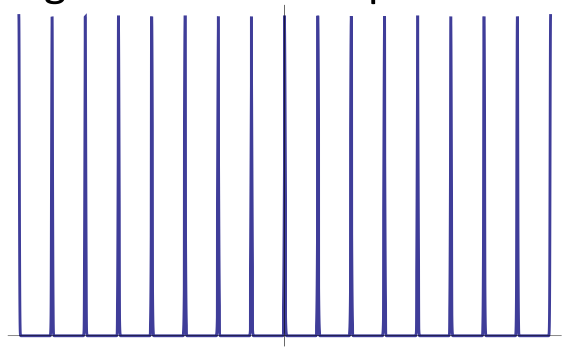
$$\exp\left[-\frac{(\omega-\omega_0)^2}{\Delta\omega^2}\right]$$

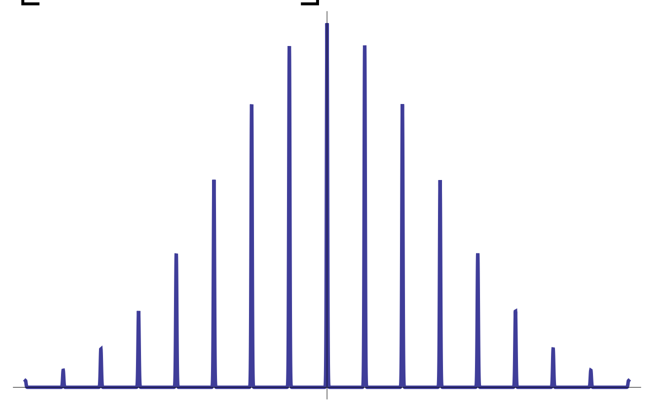
$$\mathrm{comb}(\omega/\delta\omega)$$

$$\exp\left[-\frac{(\omega-\omega_0)^2}{\Delta\omega^2}\right]\cdot\mathrm{comb}(\omega/\delta\omega)$$



Longitudinal mode spectrum

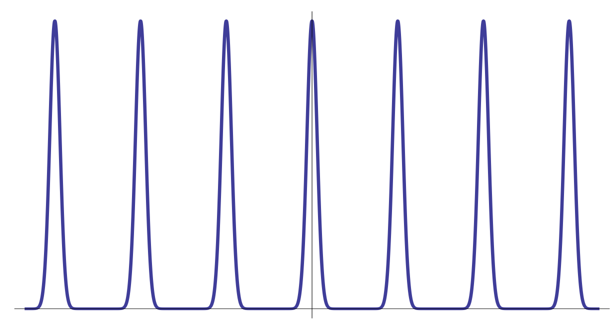




$$FT\left\{\exp\left[-\frac{(\omega-\omega_0)^2}{\Delta\omega^2}\right]\cdot\mathrm{comb}(\omega/\delta\omega)\right\}$$

$$=FT\left\{\exp\left[-\frac{(\omega-\omega_0)^2}{\Delta\omega^2}\right]\right\}\otimes\frac{1}{\delta\omega}\mathrm{comb}\left(t/(2\pi/\delta\omega)\right)$$

Pulse train output



Another calculation of the transfer function

- With this low cavity loss representation of the impulse response, FT to get to H(ω)

$$h(t)=\left(1-r^2\right)\Theta(t)e^{-t/\tau_c}\,\mathrm{comb}\left(t/T_{RT}\right)=\left(1-r^2\right)f(t)g(t)$$

$$H(\omega)=FT\left\{h(t)\right\}=\left(1-r^2\right)\frac{1}{2\pi}F(\omega)\otimes G(\omega)$$

$$F(\omega)=\int_0^\infty e^{-t/\tau_c}e^{i\omega t}\,dt=\left.\frac{e^{-t/\tau_c+i\omega t}}{-1/\tau_c+i\omega}\right|_0^\infty=\frac{1}{1/\tau_c-i\omega}$$

Complex Lorentzian

$$G(\omega)=FT\left\{\mathrm{comb}\left(t/T_{RT}\right)\right\}=\frac{2\pi}{T_{RT}}\mathrm{comb}\left(\frac{\omega}{2\pi/T_{RT}}\right)$$