

Applications of Linear Systems - Inverse Matrices - Determinants

1. Find the interpolating polynomial  $p(t) = a_0 + a_1t + a_2t^2$  for the data  $(1, 12)$ ,  $(2, 15)$ ,  $(3, 16)$ .<sup>1</sup> Noting that the system is linear in the coefficient data, we seek to find  $a_0, a_1$  and  $a_2$  that satisfies,

$$a_0 + a_1(1) + a_2(1)^2 = 12 \quad (1)$$

$$a_0 + a_1(2) + a_2(2)^2 = 15 \quad (2)$$

$$a_0 + a_1(3) + a_2(3)^2 = 16 \quad (3)$$

2. It is common to think about the equation  $\mathbf{Ax} = \mathbf{b}$  as a transformation of the vector  $\mathbf{x}$  to a new vector  $\mathbf{b}$  given by the matrix multiplication  $\mathbf{Ax}$ . In this way every matrix can be thought of as a linear transformation applied to vectors.<sup>2</sup> Probably the most common vector transformation is that of a rotation, which in  $\mathbb{R}^2$  is given by:

$$\mathbf{A} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \quad (4)$$

Let  $\mathbf{x} = [1 \ 0]^T$ . Describe or draw the results of the linear transformation  $\mathbf{Ax}$  for  $\theta \in \left\{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi\right\}$ . How would these results change if  $\mathbf{A} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$ ?

3. Given,

$$\mathbf{A} = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}.$$

Determine  $\mathbf{A}^{-1}$  via:

- (a) Calculate  $\det(\mathbf{A})$ .
- (b) The Gauss-Jordan Method (pg.317).
- (c) The cofactor representation (Theorem 2 pg.318).
- (d) Check your result by showing  $\mathbf{AA}^{-1} = \mathbf{I}$

4. Given the following for matrices:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} a+kc & b+kd \\ c & d \end{bmatrix}.$$

Calculate the determinants of the previous matrices. In each case, state the row operation used on  $\mathbf{A}$  and describe how the row operation effects the determinant.

5. The determinant has a geometric interpretation. In  $\mathbb{R}^2$ ,  $\det(\mathbf{A})$  is the area of the parallelogram formed by the two vectors  $\mathbf{a}_1, \mathbf{a}_2$ , where  $\mathbf{A} = [\mathbf{a}_1 \ \mathbf{a}_2]$ . In  $\mathbb{R}^3$ ,  $\det(\mathbf{A})$  is the volume of the parallelepiped formed by the three vectors  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ , where  $\mathbf{A} = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]$ . For an illustration of these objects please see the PDF's posted on blackboard.

Using the concept of volume, explain why the determinant of a  $3 \times 3$  matrix  $\mathbf{A}$  is zero if and only if  $\mathbf{A}$  is not invertable.<sup>3</sup>

<sup>1</sup>An interpolating polynomial for a data set is a polynomial whose graph passes through every point in the data set.

<sup>2</sup>See [http://en.wikipedia.org/wiki/Transformation\\_matrix](http://en.wikipedia.org/wiki/Transformation_matrix) for more information.

<sup>3</sup>If the three vectors,  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ , form a parallelepiped with zero volume then what can be said about their geometric configuration? If a matrix is not invertible then what can be said about the linear independence of the rows or columns?

18. We have the polynomial

$$p(t) = a_0 + a_1 t + a_2 t^2 \quad (*)$$

and the data points  $(1, 12), (2, 15), (3, 16)$ . This generates 3 linear equations,

$$p(1) = a_0 + a_1 + a_2 = 12$$

$$p(2) = a_0 + 2a_1 + 4a_2 = 15$$

$$p(3) = a_0 + 3a_1 + 9a_2 = 16$$

and the corresponding augmented matrix,

~~Q4~~

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 1 & 2 & 4 & 15 \\ 1 & 3 & 9 & 16 \end{array} \right] \xrightarrow{\substack{R2=R2-R1 \\ R3=R3-R1}} \sim \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 2 & 8 & 4 \end{array} \right] \sim$$

$$\xrightarrow{R3=R3-2R2} \sim \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 2 & -2 \end{array} \right] \xrightarrow{R3=R3/2} \sim \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & -1 \end{array} \right] \sim$$

$$\xrightarrow{R2=R2-3R3} \sim \left[ \begin{array}{ccc|c} 1 & 1 & 0 & 13 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{array} \right] \xrightarrow{R1=R1-R2} \sim \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

~~+5~~

The row equivalent linear system is then,

~~+1~~  $a_0 = 7$  which implies that,  
 $a_1 = 6$   $p(t) = 7 + 6t - t^2$   
 $a_2 = -1$  is the quadratic polynomial which interpolates  $\{t\}$ )

## Problem 2

$(\cos \theta, \sin \theta)$

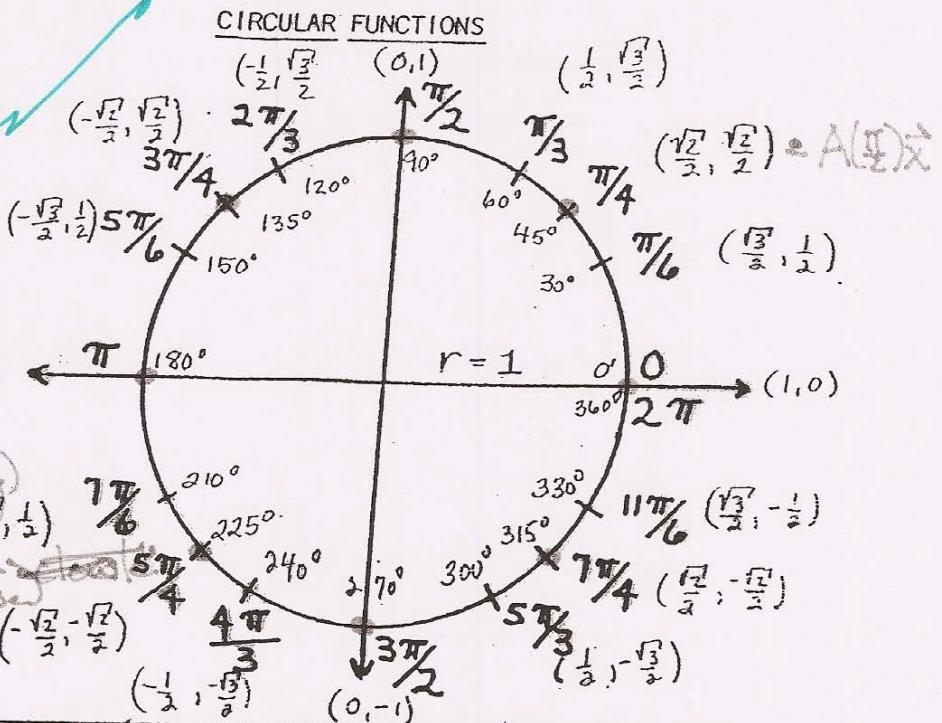
$$\text{If } \vec{x} = [1 \ 0]^T$$

then the new vectors

$$A(\theta) \vec{x}$$

will correspond to the vectors on the unit circle  $\rightarrow$ .

+5 Calculation for no of new vectors.



A Represents

+2 Rigged (norm-preserving)

~~the rotations (counter-clockwise of  $\vec{x}$ )~~

$$A = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

Rep.

Clockwise Rotations



+3

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\pi$	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	$2\pi$
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
$\tan x$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$\infty$	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$\infty$	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0
$\cot x$	$\infty$	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	$\infty$	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	$\infty$
$\sec x$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	$\infty$	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	-1	$-\frac{2}{\sqrt{3}}$	$-\sqrt{2}$	-2	$\infty$	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$\csc x$	$\infty$	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	$\infty$	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	-1	$-\frac{2}{\sqrt{3}}$	$-\sqrt{2}$	-2	$\infty$

## HOMEWORK 10

## [2] Problem 3

$$A = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

DETERMINE  $A^{-1}$  VIA(a) CALCULATE  $\det(A)$ 

$$\det(A) = 3 \det \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} - 0 \det \begin{pmatrix} 6 & 7 \\ 3 & 4 \end{pmatrix} + 2 \det \begin{pmatrix} 6 & 7 \\ 2 & 1 \end{pmatrix}$$

+ 1  $= 3(5) - 0(3) + 2(-8) = 15 - 16 = \boxed{-1}$

(b) THE GAUSS-JORDAN METHOD

$$\left[ \begin{array}{ccc|ccc} 3 & 6 & 7 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R1-3R2} \left[ \begin{array}{ccc|ccc} 3 & 0 & 4 & 1 & -3 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 3 & 2 & 2 & 0 & -3 \end{array} \right] \xrightarrow{2R1-3R2} \left[ \begin{array}{ccc|ccc} 3 & 0 & 0 & -15 & 9 & 24 \\ 0 & 2 & 0 & -4 & 4 & 6 \\ 0 & 0 & 1 & 4 & -3 & -6 \end{array} \right] \xrightarrow{2R3-3R2}$$

$$\sim \left[ \begin{array}{ccc|ccc} 3 & 0 & 4 & 1 & -3 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 4 & -3 & -6 \end{array} \right] \xrightarrow{R1-4R3} \left[ \begin{array}{ccc|ccc} 3 & 0 & 0 & -15 & 9 & 24 \\ 0 & 2 & 0 & -4 & 4 & 6 \\ 0 & 0 & 1 & 4 & -3 & -6 \end{array} \right] \div 3$$

$$\sim \left[ \begin{array}{ccc|ccc} 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 4 & -3 & -6 \end{array} \right] \xrightarrow{R2-R3} \left[ \begin{array}{ccc|ccc} 3 & 0 & 0 & -15 & 9 & 24 \\ 0 & 2 & 0 & -4 & 4 & 6 \\ 0 & 0 & 1 & 4 & -3 & -6 \end{array} \right] \div 2$$

+ 5

$$\sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 3 & 8 \\ 0 & 1 & 0 & -2 & 2 & 3 \\ 0 & 0 & 1 & 4 & -3 & -6 \end{array} \right]$$

$$A^{-1} = \boxed{\begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix}}$$

(c) THE COFACTOR REPRESENTATION.

+ 3

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} C_{11} & -C_{21} & C_{31} \\ -C_{12} & C_{22} & -C_{32} \\ C_{13} & -C_{23} & C_{33} \end{bmatrix} = \frac{1}{-1} \begin{bmatrix} 5 & -3 & -8 \\ 2 & -2 & -3 \\ -4 & 3 & 6 \end{bmatrix} = \boxed{\begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix}}$$

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## HOMEWORK 10

(d) CHECK YOUR RESULT BY SHOWING  $AA^{-1} = I$ 

$$AA^{-1} = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \quad \checkmark \quad +1$$

3

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

(a) DETERMINE THE EIGENVALUES OF A

A COFACTOR EXPANSION ALONG THE FIRST ROW GIVES:

$$(4-\lambda)(1-\lambda)^2 + 2(1-\lambda) = 0$$

$$\Rightarrow (4-\lambda)(1-\lambda) + 2 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 6 = 0 \Rightarrow (\lambda-2)(\lambda-3) = 0$$

$$\boxed{\lambda = 2, 3}$$

(b) DETERMINE THE EIGENVECTORS

$$A - 2I = \begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \begin{array}{l} R2+R1 \\ R3+R1 \end{array} \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} 2x_1 + x_3 &= 0 \\ -x_2 + x_3 &= 0 \end{aligned} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{-1}{2} \\ 1 \\ 1 \end{bmatrix}$$

IF WE CHOOSE  $x_3 = 2$ 

THE EIGENVECTOR IS:

$$\boxed{\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}}$$

Homework 5 - Solutions

Problem 4

i.  $\det(A) = ad - bc$

ii.  $\det(B) = cb - ad = -(ad - bc) = -\det(A)$

iii.  $\det(D) = d(a+kc) - c(b+kd) = ad + kdc - cb - cd \cdot k =$   
 $\stackrel{\text{Note: } k \text{ is a scalar}}{=} ad - bc = \det(A)$

iv.  $\det(C) = ad \cdot k - kc \cdot b = k(ad - bc) = k \det(A)$

+5

+5 { A  $\sim$  B by a row interchange and ii. shows  $\det(A) = -\det(B)$   
 A  $\sim$  C by a Row Scaling and iv. shows  $\det(B) = k \det(A)$   
 A  $\sim$  D by a Row Interchange where a multiple of  
 1 row is added to another. iii. shows that  $\det(A) = \det(D)$ .

2. a. Elementary matrices  $E_5, E_6$  corresponds to Row interchanges.

$\det(E_5) = -1 \cdot \det(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}) = (-1) \cdot 1 = -1$

$\det(E_6) = 1 \cdot \det(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) = 1$

The determinant of a <sup>elements</sup> row interchange matrix is -1.

b. Elementary matrices  $E_3, E_4$  correspond to Row scaling.

$\det(E_3) = k \cdot 1 \cdot 1 = k$  by diagonal matrix theorem.

$\det(E_4) = 1 \cdot k \cdot 1 = k$

The determinate of a <sup>elements</sup> row scaling by k is k.

c. Elementary matrices  $E_1, E_2$  correspond to Row Replacements.

$\det(E_1) = 1 \cdot \det(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) = 1 \cdot (1 - 0) = 1$

$\det(E_2) = 1 \cdot \det(\begin{smallmatrix} 1 & 0 \\ k & 1 \end{smallmatrix}) = 1$

The determinant of an elementary row replacement is 1.

5.

Forward direction:

Assume  $A_{3 \times 3}$  is such that  $\det(A) = 0$ . Then the volume of the parallelepiped spanned by  $\vec{a}_1, \vec{a}_2, \vec{a}_3$  has zero volume. That is, the parallelepiped does not exist. This implies that all the vectors  $\vec{a}_1, \vec{a}_2, \vec{a}_3$  lie in the same plane, and form a linearly dependent set. Thus, by the invertible matrix theorem  $A^{-1}$  does not exist.

Backward direction:

not.

Assume  $A^{-1}$  is invertible. Then the columns of  $A$  are linearly dependent and  $\text{span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$  is at most (in terms of dimension) a plane which has zero volume and cannot form a parallelepiped. Thus,  $\det(A) = 0$ .

This problem was taken from a Lin. Alg. class I taught. I did not ask the AEM students to be rigorous. I only wanted them to connect the highlighted ideas.

Grade:

+5 attempted but weak or has errors  
+10 attempted with correct connections

Note they will not talk about span or Linear Ind.

Let me know if you have questions 7/7