MATH 332 - **Linear Algebra** July 28, 2009

Homework 8, Summer 2009 **Due:** August 3, 2009

Inner-Product - Norm - Orthogonality - Gram-Schmidt - QR Factorization

1. Let,

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1}$$

Determine $\mathbf{U}$, associated with the similarity transformation $\sigma_z = \mathbf{U}\sigma_x\mathbf{U}^T$ where $\mathbf{U}$ is an orthogonal matrix. [1]

2. Prove the following:

   (a) Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Prove that $||\mathbf{u}+\mathbf{v}||^2 + ||\mathbf{u}-\mathbf{v}||^2 = 2||\mathbf{u}||^2 + 2||\mathbf{v}||^2$.

   (b) Let W be a subspace of $\mathbb{R}^n$. Prove that $W^\perp$ is a subspace of $\mathbb{R}^n$. [2]

   (c) Let $\mathbf{U}$ be an orthogonal matrix. Prove that $||\mathbf{U}\mathbf{x}|| = ||\mathbf{x}||$.

   (d) Let $\mathbf{U}_{n\times n}$ be an orthogonal matrix and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Prove that $\mathbf{U}\mathbf{x} \cdot \mathbf{U}\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$.

   (e) Let $\mathbf{U}_{n\times n}$ be an orthogonal matrix and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Prove that $\mathbf{U}\mathbf{x} \cdot \mathbf{U}\mathbf{y} = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.

3. Given,

$x \cdot y = x^T y$

$$\mathbf{y} = \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}.$$

$Ux \cdot Uy =$

$(Ux)^T Uy = x^T \underbrace{\cancel{U^T U}}_{I} y =$

$= x^T y$

   (a) Let $\mathbf{U} = [\mathbf{u}_1 \quad \mathbf{u}_2]$. Compute $\mathbf{U}^T\mathbf{U}$ and $\mathbf{U}\mathbf{U}^T$.

   (b) Let $W = \text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Compute $\text{proj}_W\, \mathbf{y}$ and $(\mathbf{U}\mathbf{U}^T)\mathbf{y}$.

   (c) Write $\mathbf{y}$ as the sum of a vector $\hat{\mathbf{y}}$ in W and a vector $\mathbf{z}$ in $W^\perp$.

   (d) Describe the geometric relationship between the plane W in $\mathbb{R}^3$ and the vectors $\hat{\mathbf{y}}$ and $\mathbf{z}$ from part **c**.

4. Given,

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}.$$

Determine the **QR** factorization of **A**.

5. In homework 6 we showed that the first four Hermite polynomials were linearly independent and thus a basis for $\mathbb{P}_3$.[3] While this makes good use of the material from 4.4 outside of the context of $\mathbb{R}^n$ it really misses the point.[4] The Hermite polynomials are orthogonal polynomials and constitute an orthonormal basis for vector space $L^2(-\infty,\infty)$.[5] To see why this is true we must define the inner-product to be,

$$f \cdot g = \int_{-\infty}^{\infty} f(x)g(x)e^{-x^2}\,dx, \tag{2}$$

[1] Consider diagonalization of σ_x and notice that its eigenvectors are orthogonal. One could hope that these might provide the columns of the U trix if they were unit-length.

[2] The procedure for this proof is outlined in Lay (problem 6.1.30 pg.383).

[3] We take without proof that the first $n+1$ Hermite polynomials are linearly independent and thus a basis for $\mathbb{P}_n$.

[4] The Hermite polynomials are prevalent in statistics, applied mathematics and physics but not in the context of polynomial spaces.

[5] The vector space $L^2(-\infty,\infty)$ is an infinite dimensional complete inner-product space or a Hilbert space, in honor of David Hilbert http://en. kipedia.org/wiki/David_Hilbert. The space L^2, which is an abstraction of standard Euclidean space, is important because its elements must have ite length and any infinite-sequence of elements must converge to a point in L^2. The condition that 'vectors' must have finite length typically implies at they have finite energy, which is what one would hope. While, the convergence properties allows use to take limits without leaving the space.[6]

which is different than our standard definition in $\mathbb{R}^n$. [8] We take without proof that this definition satisfies the axioms of an inner-product. Recall the first few Hermite Polynomials, [9]

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = -2 + 4x^2, \quad H_3(x) = -12x + 8x^3, \quad x \in (-\infty, \infty),$$

satisfying the Rodrigues representation,

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}. \tag{3}$$

(a) Prove that $H_{2n}(x)$ is and even function and that $H_{2n+1}(x)$ is an odd function. [10]

(b) Prove that the even Hermite polynomials are orthogonal to the odd Hermite polynomials.

(c) Normalize H_0 and H_1.

(d) Using the normalized Hermite polynomials apply Gram-Schmidt and find $H_2(x)$. [11]

$$\cancel{\tfrac{d}{dx}} = \frac{df(-x)}{dx} = \frac{df(x)}{dx}$$

$$\left.\frac{df}{dx}\right|_{x=-x} = \frac{df}{dx} = \frac{d}{dx}[f(-x)] = -\frac{df}{dx}$$

$g(x) = f'(x)$ then $g(-x) = -f'(x) = -g(x)$ which is odd

$$g(-x) = \left.\frac{df}{dx}\right|_{x=-x} = \frac{d}{dx}[f(-x)] = -\frac{df}{dx} = -g(x)$$

[6]Indeed, things would be very bad if this were not the case. Consider the infinite sum, $\sum_{n=0}^{\infty} \frac{4(-1)^n}{2n-1}$. The summands are all rational but this sum nverges to π, which is irrational. That is, the rationals are not closed under limits of arbitrary linear combinations![7]

[7]Yeah, I footnoted a footnote. What of it?!

[8]If we used the standard inner-product and made the Hermite polynomials an orthonormal basis, via GramSchmidt, for $\mathbb{P}^n$ then we would have tten to the standard polynomial basis, which is nothing new.

[9]For more we can look at http://en.wikipedia.org/wiki/Hermite_polynomials. There are, in general, infinitely-many of them arising as eigen- nctions of the differential operator $\frac{d^2}{dx^2} - x\frac{d}{dx}$.

[10]Recall that an even function has the property that $f(-x) = f(x)$ and an odd function has the property that $f(-x) = -f(x)$.

[11]MIT's open courseware site has a nice discussion of GS applied to the Legendre polynomials. web.mit.edu/18.06/www/Spring09/legendre.pdf To this first consider a general quadratic, $H_2(x) = ax^2 + bx + c$, and argue that $b = 0$. Next, we want to find a and c such that $H_2(x)$ is orthogonal $H_1(x)$ and $H_0(x)$. Gram-Schmidt gives us a formula for this, page 404 of the text, only every inner-product must be thought of in the sense of (2). ter this calculation you should have a relation between a and c. To find a normalize $H_2(x)$ and compare your result to $H_2(x)$ as it is given. They ould look the same up a multiplicative constant.

1.

Diagonalize $\bar{S}_x$.

$$\bar{S}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \det(\bar{S}_x - \lambda I) = \lambda^2 - 1 = 0$$
$$\Rightarrow \lambda = \pm 1$$

Case $\lambda = 1$

$$\bar{S}_x - \lambda I = 0 \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = -x_2 \\ x_2 \text{ is free} \end{array}$$

$$\Rightarrow \quad \bar{x} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix}, \text{ Choose } x_2 = \frac{1}{\sqrt{2}} \text{ to normalize } \bar{x}.$$

$$\Rightarrow \quad \bar{x}^{(1)} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad \|\bar{x}^{(1)}\| = 1$$

Case $\lambda = -1$

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \Rightarrow \bar{x}^{(2)} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix}, \text{ Choose } x_2 = 1/\sqrt{2}$$

which gives,

$$\bar{x}^{(2)} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

This gives the diagonalization for $\bar{S}_x$ as,

$$\bar{S}_x = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \left\{ \frac{1}{-1} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \right\}$$

Noting that the diagonal matrix on $\bar{S}_z$ are the same and that $P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = P^T = P^{-1}$ implies that,

$$P^{-1} \bar{S}_x P = P^{-1} P \bar{S}_z P^{-1} P = \bar{S}_z \quad \text{which means}$$

$$U = P^{-1} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = P^T = U^T.$$

1. Let $u, v \in \mathbb{R}^n$ then

$\|u+v\|^2 + \|u-v\|^2 = (u+v)^T(u+v) + (u-v)^T(u-v) =$

$= (u^T+v^T)(u+v) + (u^T-v^T)(u-v) =$

$= u^Tu + v^Tv + 2u^Tv + u^Tu + v^Tv - 2u^Tv =$

$= 2\|u\|^2 + 2\|v\|^2 \Rightarrow \|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$

2. Let W be a subspace of $\mathbb{R}^n$. Also let $z_1, z_2 \in W^\perp \subset \mathbb{R}^n$, $c \in \mathbb{R}$

Then for any $u \in W$ we have

$(cz_1 + z_2)^T u = cz_1^T u + z_2^T u = c \cdot 0 + 0 = 0$

which implies that $cz_1 + z_2 \in W^\perp$. This shows that $W^\perp$ is closed under both vector addition and scalar multiplication.

also, Note that $0 \in W^\perp$ since $\vec{0}^T u = 0$. Thus $W^\perp$ is a subspace $\mathbb{R}^n$.

3. a. Since

$\vec{u}_1^T \vec{u}_2 = -1 + 1 = 0$, $\vec{u}_1^T \vec{u}_3 = 2 - 2 = 0$, $u_2^T u_3 = -2 + 4 - 2 = 0$

and since $\vec{u}_i^T \vec{u}_j = u_j^T u_i$, we have that $S = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal set. Since orthogonal vectors are linearly independent and 3 linearly independent vectors in $\mathbb{R}^3$ spans $\mathbb{R}^3$, S forms an orthogonal basis for $\mathbb{R}^3$.

1/2

2. ~~If the columns of U are orthogonal then they are necessarily independent. Since $U_{n\times n}$ has n-many linearly independent columns, U^{-1} exists.~~

3, a. $\|U\vec{x}\|^2 = (U\vec{x})^T(U\vec{x}) = x^T U^T U \vec{x} = \vec{x}^T\vec{x} \quad \|\vec{x}\|^2 \iff$
$\iff \|U\vec{x}\| = \|\vec{x}\|$

b. $(U\vec{x})^T(U\vec{y}) = \vec{x}^T U^T U\vec{y} = \vec{x}^T\vec{y}$

c. Assume $(U\vec{x})^T(U\vec{y}) = 0$, by part b. $\vec{x}^T\vec{y}$ is zero.
Assume $\vec{x}^T\vec{y} = 0$ then by part b. $(Ux)^T(Uy) = 0$.

4. a. $U = \begin{bmatrix} 2/3 & -2/3 \\ 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix}$, $U^TU = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix}\begin{bmatrix} 2/3 & -2/3 \\ 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$UU^T = \begin{bmatrix} 2/3 & -2/3 \\ 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 8/9 & -2/9 & 2/9 \\ -2/9 & 5/9 & 4/9 \\ 2/9 & 4/9 & 5/9 \end{bmatrix}$$

b. Note $U_1^T U_2 = -\frac{4}{9} + \frac{2}{9} + \frac{2}{9} = 0$
Thus $\vec{u}_1, \vec{u}_2$ form an orthogonal basis for span$\{\vec{u}_1, \vec{u}_2\}$
note also that $\vec{u}_1^T\vec{u}_1 = 1$, $\vec{u}_2^T\vec{u}_2 = 1$. Thus,

$$\text{Proj}_W\vec{y} = y^T\vec{u}_1\vec{u}_1 + y^T\vec{u}_2\vec{u}_2 = 6\cdot\begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix} + 3\begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

$$UU^T\vec{y} = \begin{bmatrix} 8/9 & -2/9 & 2/9 \\ -2/9 & 5/9 & 4/9 \\ 2/9 & 4/9 & 5/9 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

Thus $\text{proj}_W \vec{y} = UU^T\vec{y}$.

c. $\vec{y} = \hat{y} + \vec{z} \iff \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -4 \end{bmatrix} = \vec{z}$

Note $\vec{z}^T \hat{y} = 20 - 20 = 0$.

$$\vec{y} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ -4 \end{bmatrix}$$

d. $W = \text{Span}\{\vec{u}_1, \vec{u}_2\}$ is a plane in $\mathbb{R}^3$. The vector $\hat{y}$ is the closest point on the plane to $y \in \mathbb{R}^3$. $\vec{z}$ is the vector in $W^\perp$ that has $\vec{y}$ at it's tip and $\hat{y}$ ~~at~~ at it's tail, which is orthogonal to W.

1. $A = \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}$

Let $\vec{a}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} 2 \\ 1 \\ 4 \\ -4 \\ 2 \end{bmatrix}$, $\vec{a}_3 = \begin{bmatrix} 5 \\ -4 \\ -3 \\ 7 \\ 1 \end{bmatrix}$

$B = \{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ is a basis for Col A.

Use Gram-Schmidt to construct an orthonormal basis for Col A.

$\vec{u}_1 = \vec{a}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$

$\vec{u}_2 = \vec{a}_2 - \dfrac{\vec{a}_2^T \vec{u}_1}{\vec{u}_1^T \vec{u}_1} = \begin{bmatrix} 2 \\ 1 \\ 4 \\ -4 \\ 2 \end{bmatrix} - \dfrac{2-1-4-4+2}{5} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix}$

$\vec{u}_3 = \vec{a}_3 - \dfrac{\vec{a}_3^T \vec{u}_2}{\vec{u}_2^T \vec{u}_2} \vec{u}_2 - \dfrac{\vec{a}_3^T \vec{u}_1}{\vec{u}_1^T \vec{u}_1} \vec{u}_1 = \begin{bmatrix} 5 \\ -4 \\ -3 \\ 7 \\ 1 \end{bmatrix} - \dfrac{15-9-21+3}{36} \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix} - \dfrac{5+4+3+7+1}{5} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} =$

$= \begin{bmatrix} 5 \\ -4 \\ -3 \\ 7 \\ 1 \end{bmatrix} + \dfrac{1}{3} \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix} - 4 \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -2 \end{bmatrix}$

Since $B_1 = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal basis for Col A,

$$B_1' = \{\vec{q}_1, \vec{q}_2, \vec{q}_3\}$$

where,

$$\vec{q}_1 = \frac{1}{\|\vec{u}_1\|}\vec{u}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{q}_2 = \frac{1}{\|\vec{u}_2\|}\vec{u}_2 = \frac{1}{6}\begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$$\vec{q}_3 = \frac{1}{\|\vec{u}_3\|}\vec{u}_3 = \frac{1}{4}\begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

is an orthonormal basis for Col A.

Thus the matrix Q is then $Q = [\vec{q}_1\ \vec{q}_2\ \vec{q}_3]$ and the mat. R is given by,

$$R = Q^T A = \begin{bmatrix} 1/\sqrt{5} & -1/\sqrt{5} & -1/\sqrt{5} & 1/\sqrt{5} & 1/\sqrt{5} \\ 1/2 & 0 & 1/2 & -1/2 & 1/2 \\ 1/2 & 0 & 1/2 & 1/2 & -1/2 \end{bmatrix}_{3\times 5} \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}_{5\times 3} = \begin{bmatrix} 5/\sqrt{5} & -5/\sqrt{5} & 20/\sqrt{5} \\ 0 & 6 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

Gram-Schmidt and Hermite Poly:

$H_0(x) = 1, \quad H_1(x) = 2x$

a) Note that:

i) If $f(x)$ is even then $f(-x) = f(x)$ and by the chain Rule,

$$g(x) = \frac{df}{dx} \quad \text{while} \quad g(-x) = \frac{d}{dx}[f(-x)] = -\frac{df}{dx} = -g(x)$$

which implies that $\frac{df}{dx}$ is odd.

A similar Result is true for the derivative of an odd fxn.

Now consider,

$$H_{2n}(x) = (-1)^{2n} e^{x^2} \frac{d^{2n}}{dx^{2n}} e^{-x^2} = e^{x^2} \frac{d^{2n}}{dx^{2n}} e^{-x^2}$$

Since e^{-x^2} is even $\frac{d^{2n}}{dx^{2n}} e^{-x^2}$ is also even. Then

$$H_{2n}(x) = e^{x^2} \cdot \text{even fxn of } x \equiv \text{Even fxn of } x.$$

$\Rightarrow H_{2n}(x)$ is Even.

A similar argument shows that $H_{2n+1}(x)$ is odd.

b) $\int_{-\infty}^{\infty} H_{2n}(x)\, H_{2n+1}(x)\, e^{-x^2}\, dx = 0 \Rightarrow H_{2n}$ is orthogonal to H_{2n+1}.

<u>Why?</u>: H_{2n}, e^{-x^2} are Even fxn

H_{2n+1} ~~an~~ is an odd fxn

The overall symmetry of the integrand is odd, and the integral of an odd fxn over all space is zero.

c) $\|H_0(x)\| = \sqrt{\langle H_0 | H_0 \rangle} = \left(\int_{-\infty}^{\infty} H_0(x) H_0(x)\, e^{-x^2}\, dx \right)^{1/2} =$

$= \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx \right)^{1/2} = \sqrt[4]{\pi}$

$\|H_1(x)\| = \sqrt{\langle H_1, H_1 \rangle} = \left(\int_{-\infty}^{\infty} 4x^2 e^{-x^2}\, dx \right)^{1/2} = 2\left[\cancel{-\frac{x}{2} e^{-x^2}}\Big|_{-\infty}^{\infty} + \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2}\, dx \right]^{1/2}$

u	dv
x	$x e^{-x^2}$
1	$-\frac{1}{2} e^{-x^2}$

$= 2\left[\frac{\sqrt{\pi}}{2} \right]^{1/2} = 2^{1/4}\sqrt[4]{\pi} = \sqrt[4]{4\pi}$

$\Rightarrow H_0(x) = \frac{1}{\sqrt[4]{\pi}}, \quad H_1(x) = \frac{\sqrt{2}}{\sqrt[4]{\pi}}\, x$

d) Assume,

$$H_2(x) = \alpha x^2 + \beta x + \gamma$$

Since H_2 is even $\beta = 0 \Rightarrow H_2(x) = \alpha x^2 + \gamma$

Application of G.S. gives,

$$V_2 = H_2 - \frac{H_2 \cdot H_0}{H_0 \cdot H_0} H_0 - \cancel{\frac{H_2 \cdot H_1}{H_1 \cdot H_1}}^{0} H_1 =$$

$$= H_2 - \frac{H_2 \cdot H_0}{H_0 \cdot H_0} H_0$$

Where

$$H_2 \cdot H_0 = \langle H_2, H_0 \rangle = \int_{-\infty}^{\infty} (\alpha x^2 + \gamma) e^{-x^2} dx =$$

$$= \alpha \int_{-\infty}^{\infty} x^2 e^{-x^2} dx + \gamma \int_{-\infty}^{\infty} e^{-x^2} dx = \alpha \frac{\sqrt{\pi}}{2} + \gamma \sqrt{\pi}$$

and

$$\langle H_0, H_0 \rangle = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

$\Rightarrow$

$$V_2 = H_2 - \left(\frac{\alpha}{2} + \gamma\right) H_0 = \alpha x^2 + \gamma - \left(\frac{\alpha}{2} + \gamma\right) 1 =$$

$$= \alpha x^2 - \frac{\alpha}{2}$$

What is α^2?

$$\|H_2(x)\| = \sqrt{\langle H_2, H_2\rangle} = \left(\int_{-\infty}^{\infty}\left(\alpha x^2 - \frac{\alpha}{2}\right)^2 e^{-x^2}dx\right)^{1/2} =$$

$$= \left[\int_{-\infty}^{\infty}\alpha^2 x^4 e^{-x^2}dx - \int_{-\infty}^{\infty}\alpha^2 x^2 e^{-x^2}dx + \frac{\alpha^2}{4}\int_{-\infty}^{\infty}e^{-x^2}dx\right]^{1/2} =$$

$$= \left[\alpha^2\int_{-\infty}^{\infty}x^4 e^{-x^2}dx - \alpha^2\frac{\sqrt{\pi}}{2} + \frac{\alpha^2}{4}\sqrt{\pi}\right]^{1/2} =$$

Where

$$\int_{-\infty}^{\infty}x^4 e^{-x^2}dx = \cancel{-\frac{x^3}{2}e^{-x^2}\Big|_{-\infty}^{\infty}}^{\,0} + \frac{3}{2}\int_{-\infty}^{\infty} \ldots$$

$u = x^3 \quad dv = xe^{-x^2}$

$du = 3x^2dx \quad v = -\frac{1}{2}e^{-x^2}$

$$= \frac{3}{2}\cdot\frac{\sqrt{\pi}}{2}$$

$$= \left[\alpha^2\cdot\frac{3}{2}\cdot\frac{\sqrt{\pi}}{2} - \alpha^2\frac{\sqrt{\pi}}{2} + \alpha^2\frac{\sqrt{\pi}}{4}\right]^{1/2} = 1$$

$$\Rightarrow \frac{3\alpha^2\sqrt{\pi}}{4} - \frac{\alpha^2\sqrt{\pi}}{2} + \frac{\alpha^2\sqrt{\pi}}{4} =$$

$$= \frac{\alpha^2\sqrt{\pi}}{2} = 1 \Rightarrow \alpha^2 = \frac{2}{\sqrt{\pi}} \Rightarrow \alpha = \frac{\sqrt{2}}{\sqrt[4]{\pi}}$$

and

$$H_2(x) = \frac{\sqrt{2}}{\sqrt[4]{\pi}}x^2 - \frac{1}{\sqrt[4]{4\pi}}$$ is the normalized quadratic hermite poly.