Derivations of integral formulae for second-order linea ODE. 9/3/2012

Given,

$$a(x)\, y'' + b(x)\, y' + c(x)\, y = f(x) \quad (1)$$

and let $y_1(x)$ be 1 soln to its homogeneous case. To find a second soln we could hope

$$y_2(x) = k(x)\, y_1(x), \quad (2)$$

which is to say that y_2 must be more than just a multiple of y_1. We, of course, check (2) against (1) by

$$y_2'(x) = k' y_1 + k y_1'$$

$$y_2''(x) = k'' y_1 + k' y_1' + k' y_1' + k y_1''$$

to get [omit the x-dep to save Room]

$$a\left[k'' y_1 + 2k' y_1' + \underline{k y_1''}\right] + b\left[k' y_1 + \underline{k y_1'}\right] + \underline{c k y_1} = 0$$

Notice that the blue tems sum to

$$aky_1'' + bky_1' + cky_1 = k\underbrace{(ay_1'' + by_1'' + cy_1)}_{\text{Since } y_1 \text{ solves (1) when } f(x)=0 \text{ this term is zero}} = 0$$

This leaves us with

$$(ak'' + bk')y_1 + 2k'ay_1' = 0$$

which can be written as,

$$(by_1 + 2ay_1')k' + ayk'' = 0$$

Assuming $u = k' \Rightarrow u' = k$ & and

$$(by_1 + 2ay_1')u + ay_1u' = 0,$$

which is linear first order in u. Solving this gives

$$u(x) = e^{-\int\left[\frac{b}{a} + \frac{2y_1'}{y}\right]dx}$$

$$= e^{-\int \frac{b}{a}dx + \ln(y^2)} = \frac{e^{-\int \frac{b}{a}dx}}{y^2} \Rightarrow k(x) = \int \frac{p(x)}{y^2(x)}dx$$

$$p(x) = e^{-\int \frac{b}{a}dx}$$

This was not clear. Redo.

We have

$$(by_1 + 2ay_1')u + ay_1 u' = 0, \quad \begin{matrix} u' = k'' \\ u = k' \end{matrix}$$

$$\Rightarrow \quad \frac{u'}{u} = \frac{-by_1 - 2ay_1'}{ay_1}$$

$$\Rightarrow \int \frac{u'}{u}dx = \int \frac{du}{u} = \ln|u| = \int \left(-\frac{b}{a} - \frac{2y_1'}{y_1}\right)dx$$

$$= -\int \frac{b}{a}dx - 2\int \frac{y_1'}{y_1}dx = -\int \frac{b}{a}dx - 2\int \frac{dy_1}{y_1} =$$

$$= -\int \frac{b}{a}dx - 2\ln|y_1|$$

$$\Rightarrow \quad u(x) = e^{-\int \frac{b}{a}dx + \ln(y_1^{-2})} = e^{-\int \frac{b}{a}dx} e^{\ln(y_1^{-2})}$$

$$= \frac{e^{-\int \frac{b}{a}dx}}{y_1^2} \Rightarrow k = \int u\,dx = \int \frac{p(x)}{[y_1(x)]^2}dx$$

Where $p(x) = e^{-\int \frac{b(x)}{a(x)}dx}$

5.1: Let $a(x)=1$, $b(x)=4$, $c(x)=4$, $f(x)=e^{-2x}$

$$y''+4y'+4y=e^{-2x}$$

Homogeneous Problem: $y_1(x)=e^{-2x}$ is one soln

then

$$p(x)=e^{-\int \frac{b(x)}{a(x)}dx}=e^{-\int \frac{4}{1}dx}=e^{-4x+c}=e^{c}e^{-4x}=\tilde{c}e^{-4x},\quad \tilde{c}\in\mathbb{R}^+$$

and

$$k(x)=\int \frac{\tilde{c}e^{-4x}}{\left[e^{-2x}\right]^2}dx=\tilde{c}x+d,\quad d\in\mathbb{R}$$

Thus,

$$y_2(x)=k(x)y_1(x)=(\tilde{c}x+d)e^{-2x}$$

$$=\tilde{c}xe^{-2x}+\underbrace{de^{-2x}}_{\text{more } y_1(x)!}$$

~~Thus~~ Keeping only independent terms gives

$$y_2(x)=xe^{-2x}$$

$$\Rightarrow y_h(x)=C_1e^{-2x}+C_2xe^{-2x},\quad C_1,C_2\in\mathbb{R}$$

<u>Variation of Parameters</u>: It is possible, using the Homogeneous soln, to define the particular soln. The idea is similar to the previous assumption (2). In this case

$$y_p(x) = k_1(x)\,y_1(x) + k_2(x)\,y_2(x),$$

which says that ~~if~~ we hope the Particular soln is just a variation of the homogeneous soln.

$$y_p'(x) = k_1' y_1 + k_1 y_1' + k_2' y_2 + k_2 y_2'$$

$$y_p''(x) = k_1'' y_1 + k_1' y_1' + k_1' y_1' + k_1 y_1'' + k_2'' y_2 + k_2' y_2' + k_2' y_2' + k_2 y_2''$$

which then gives from (1)

Purple terms are Soln to homogeneous problem. Sum to zero.
Red terms " ". Sum to zero.

$$a\left[k_1'' y_1 + 2k_1' y_1' + k_1 y_1'' + k_2'' y_2 + 2k_2' y_2' + k_2 y_2''\right] +$$

$$+ b\left[k_1' y_1 + k_1 y_1' + k_2' y_2 + k_2 y_2'\right] + c\left[k_1 y_1 + k_2 y_2\right] =$$

$$= a\frac{d}{dx}\left[k_1' y_1\right] + a k_1' y_1' + a\frac{d}{dx}\left[k_2' y_2\right] + a k_2' y_2' + b\left[k_1' y_1 + k_2' y_2\right] = f(x)$$

Note that we are looking for k_1, k_2 but have only 1 Equ. One last rewrite gives

$$a \frac{d}{dx}\left[k_1' y_1 + k_2' y_2 \right] + b\left[k_1' y_1 + k_2' y_2 \right] + a k_1' y_1' + a k_2' y_2' = f(x)$$

If we require $k_1' y_1 + k_2' y_2 = 0$ then the Equ becomes,

$$k_1' y_1' + k_2' y_2' = f(x)$$

Which are 2 Equ on the k_1', k_2' variables. Again

$$k_1' y_1 + k_2' y_2 = 0$$
$$a k_1' y_1' + a k_2' y_2' = f(x)$$

The soln to such a system can be found many ways to get

$$k_1' = -\frac{y_2 f(x)}{a(x) W(x)}, \quad k_2' = \frac{y_1 f(x)}{W(x)}$$

where $W(x) = y_1 y_2' - y_2 y_1'$

After an integration step we get

$$k_1(x) = -\int \frac{y_2 f(x)}{a(x)\, W(x)}\, dx$$

$$k_2(x) = \int \frac{y_1 f(x)}{a(x)\, W(x)}\, dx$$

and

$$y_p(x) = -y_1 \int \frac{y_2(x) f(x)}{a(x)\, W(x)}\, dx + y_2 \int \frac{y_1(x) f(x)}{a(x)\, W(x)}\, dx$$

* Key Point: If you have 1 sol$\underline{n}$ to a second order linear ODE then it is possible to construct the rest via integration.

However, it is often easier to just guess in the simple cases

5.2: Given

$$y'' + 4y' + 4y = e^{-2x}$$

Where

$$y_h(x) = C_1 y_1(x) + C_2 y_2(x), \quad y_1(x) = e^{-2x}$$
$$y_2(x) = xe^{-2x}$$

$$W(x) = e^{-2x}\left(e^{-2x} - 2xe^{-2x}\right) - (-2)e^{-2x}\left(xe^{-2x}\right) =$$

$$= e^{-4x} - 2xe^{-4x} + 2xe^{-4x} = e^{-4x}$$

Thus,

$$y_p(x) = y_2(x)\int \frac{e^{-2x} \cdot e^{-2x}}{e^{-4x}}dx - y_1(x)\int \frac{xe^{-2x}e^{-2x}}{e^{-2x}}dx =$$

$$= y_2(x)\left[x + C_3\right] - y_1(x)\left[\frac{x^2}{2} + C_4\right] =$$

$$= \frac{x^2 e^{-2x}}{2} + \underbrace{y_2(x)C_3 - y_1(x)C_4}_{\text{This is just more homogeneous sols}} \Rightarrow y_p(x) = \frac{x^2 e^{-2x}}{2}$$

5.2: Undetermined Coeff.

Since $f(x) = e^{-2x}$ and $y_1(x) = e^{-2x}$

$y_2(x) = xe^{-2x}$

we guess,

$$\cancel{f(x)}\; y_p(x) = Ax^2e^{-2x}$$

$$\Rightarrow \quad y_p'(x) = 2Axe^{-2x} - 2Ax^2e^{-2x}$$

$$y_p''(x) = 2Ae^{-2x} - 4Axe^{-2x} - 4Axe^{-2x} + 4Ax^2e^{-2x}$$

and

$$y_p'' + 4y_p' + 4y_p = 2Ae^{-2x} \cancel{- 8Axe^{-2x}} \cancel{+ 4Ax^2e^{-2x}} +$$

$$+ \cancel{8Axe^{-2x}} \cancel{- 8Ax^2e^{-2x}}$$

$$\cancel{+ 4Ax^2e^{-2x}} = 2Ae^{-2x} = e^{-2x} = f(x)$$

0 0

$$\Rightarrow \quad A = \frac{1}{2} \Rightarrow y_p(x) = \frac{x^2e^{-2x}}{2}$$

5.3: Problem where you must use the integral formulae

$$y'' + y = \sec(x)$$

$$\Rightarrow \quad y_h(x) = C_1 \cos(x) + C_2 \sin(x)$$

$$W(x) = \cos(x)\cos(x) - (-)\sin(x)\sin(x) = 1$$

$$y_p(x) = y_2 \int \frac{\sec(x)\cdot\cos(x)}{1}\,dx - y_1 \int \frac{\sec(x)\sin(x)}{1}\,dx$$

$$= y_2 \int dx - y_1 \int \tan(x)\,dx =$$

$$= y_2(x)\cdot\left[x + c_1\right] - y_1(x)\left[\ln|\cos(x)| + c_2\right] =$$

$\cos(x)$

$$= x\sin(x) + \ln|\sec(x)| + \text{homogeneous stuff.}$$