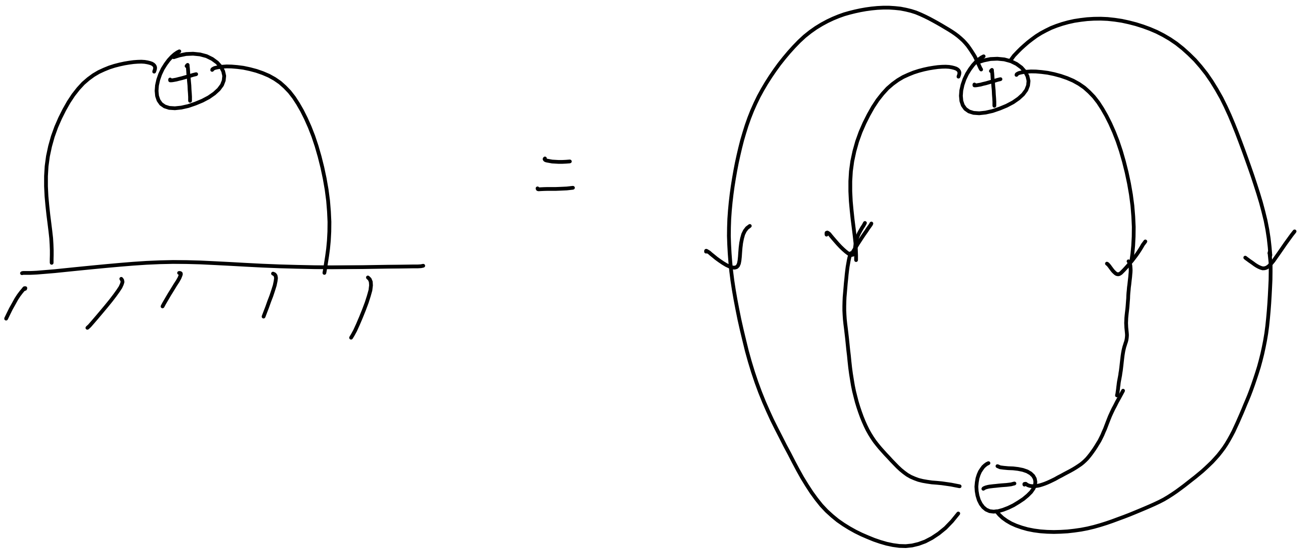
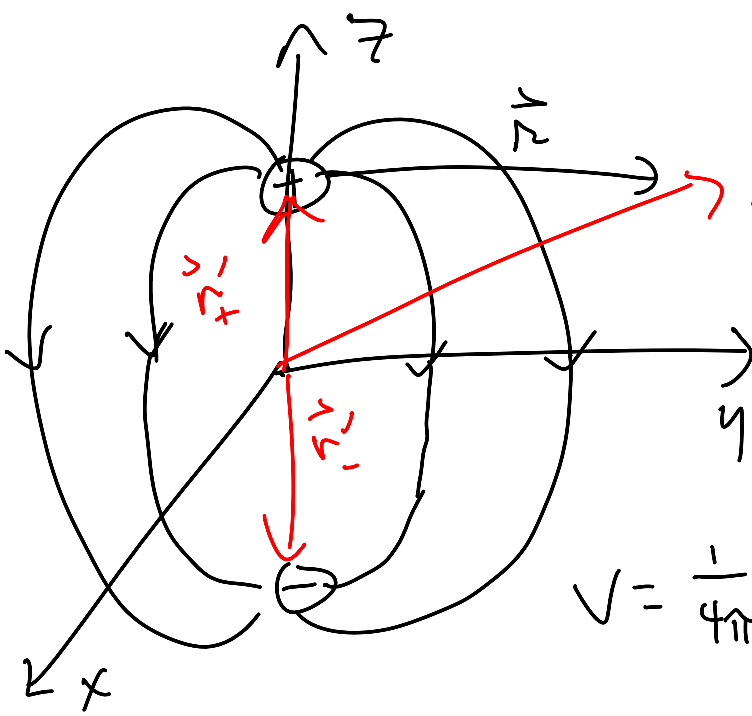
Homework 12 solutions

Homework problem 1.) Sketch the charge density as a function of position for a charge Q a distance D above an infinite conductor.

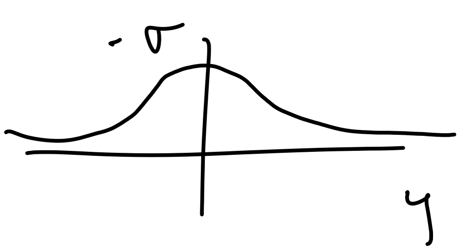




$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$V = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{|\vec{r}_+|} - \frac{Q}{|\vec{r}_-|}\right)$$

$$\vec{E} = -\vec{\nabla}V\Big|_{z=0} = \frac{\sigma}{\epsilon_0}$$



$$EA = \frac{\sigma}{\epsilon_0}A; \quad \sigma = \epsilon_0 E$$

Homework problem 2.) A thermonuclear weapon was detonated underground 30 years ago. The radioactive lifetime of the remants is thousands of years so the heat energy generated can be accurately modeled as being steady. It can be shown that the PDE for the steady state temperature distribution in the Earth is

$$k\nabla^2 T = g$$

where k is the thermal conductivity and g is the heat source energy per volume.

$$\nabla^2 V = -\rho/\epsilon_0$$

This is Poisson's equation. We can model the surface of the Earth above the detonation as not allowing any thermal energy to flow out of it. The parameters in electrostatics V and E are the analogs of T and g where the flow of heat energy per time per area h is

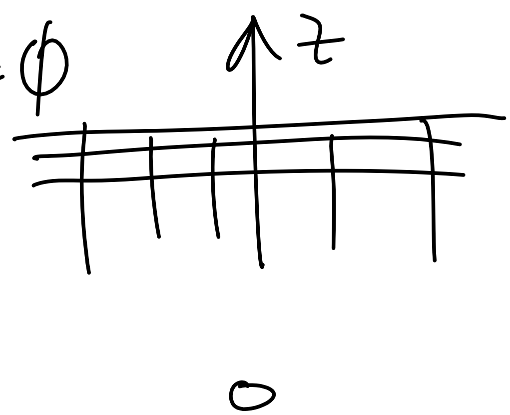
$$\vec{h} = -k\vec{\nabla}T$$

$$\vec{E} = -\vec{\nabla}V$$

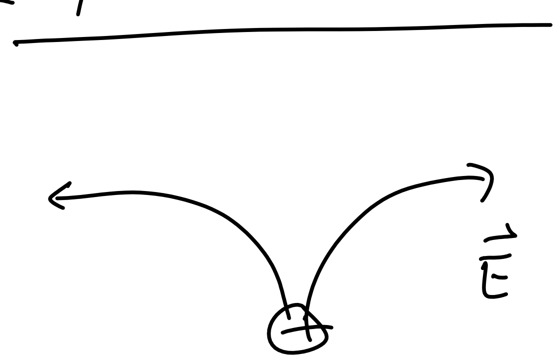
The boundary condition for the heat problem is that no flow of heat energy per second from the surface of the Earth. This is analogous to no component of the electric field perpendicular to a similar surface.

$\oplus$

$$\frac{\partial V}{\partial z} = \phi \qquad E_\perp = \phi \qquad \vec{E} = -\vec{\nabla}V$$

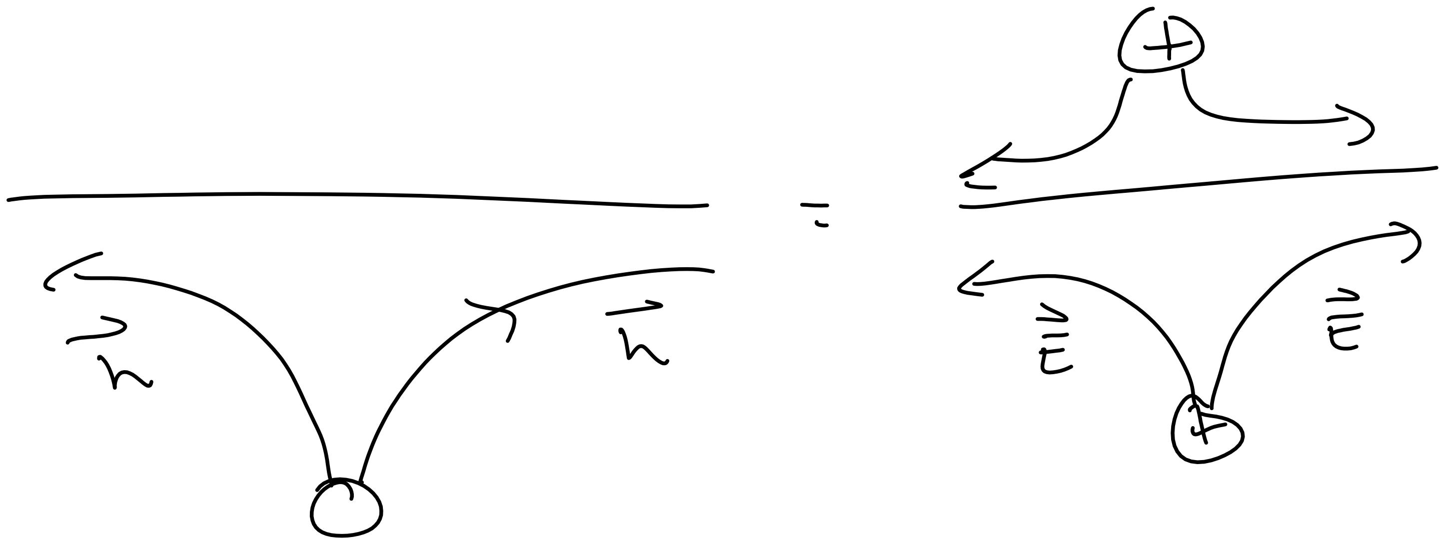


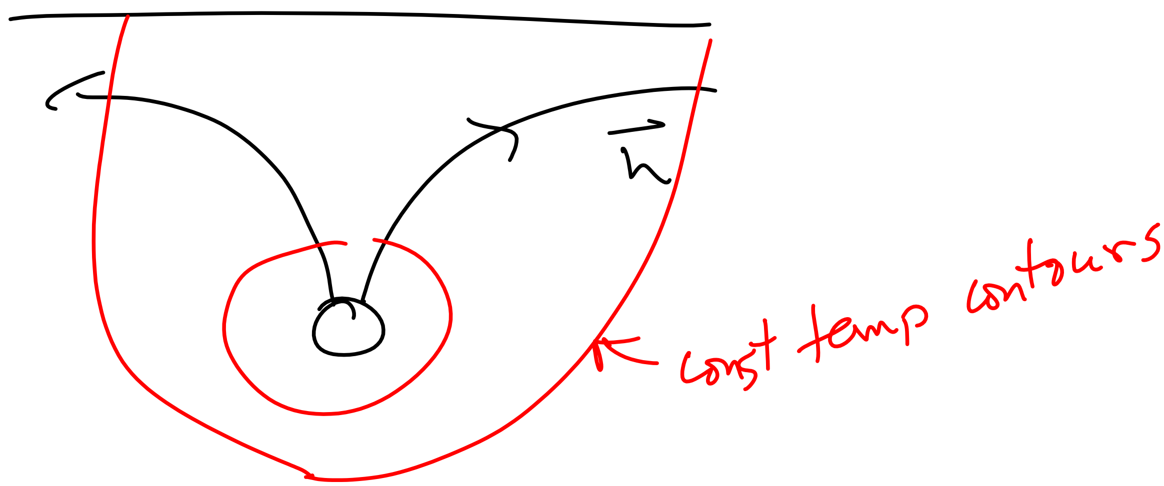
heat source

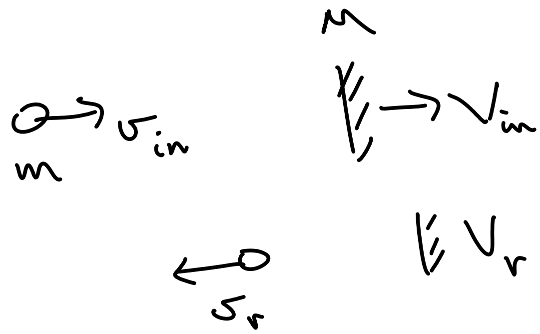


$$\nabla^2 T = g/k \qquad \nabla^2 V = -\rho/\epsilon_0$$

Set up relaxation spreadsheet. Put 0 on side and bottom bndries. For the top row put eqn that value of cell one cell below is the value of that top row cell.




const temp contours



3.) Probability density $= \psi\psi^* = P$

$$\psi_{in} = e^{ik_{in}x - \omega_{in}t} \; ; \quad P_{incident} = 1$$

$$\psi_{reflected} = e^{ik_{reflected}x - \omega t} \qquad p = mv_r = \hbar k_r$$

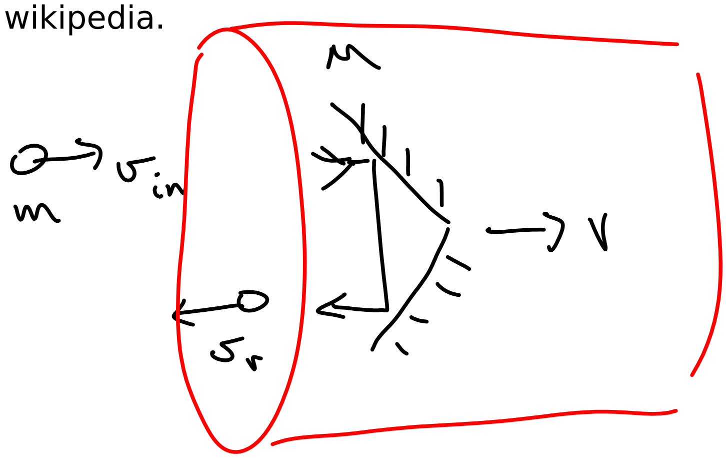
$$\left.\begin{aligned} mv_{in} + MV &= mv_r + MV_r \\ \tfrac{1}{2}mv_{in}^2 + \tfrac{1}{2}MV^2 &= \tfrac{1}{2}mv_r^2 + \tfrac{1}{2}MV_r^2 \end{aligned}\right\} \text{2 eqns in } v_r \text{ \& } V_r$$

Reflection from a massive mirror yields

$$v_r = 2V_{in} - v_{in}$$

$$k_r = \frac{m(2V_{in} - v_{in})}{\hbar}$$

To eliminate issues of interference between the incident and reflected beams,
let the mirror (corner reflector) displace the reflected beam.
Look up corner reflector on wikipedia.



$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \qquad \rho = \psi \psi^*$$

$$\vec{J} = \frac{\hbar}{2mi}\left(\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*\right)$$

$\frac{\partial}{\partial x}$ in 1-D

$$\downarrow$$

$$\oint \vec{J} \cdot d\vec{a} = -\frac{\partial}{\partial t} \int \rho \, d\tau$$

dx

prob / length

$$J_{in} = e^{-ik_{in}x} \frac{\hbar i k_{in}}{2mi} e^{ik_{in}x} - e^{ik_{in}x} \frac{(-i\hbar k_{in})}{2mi} e^{-ik_{in}x}$$

$$= \frac{2\hbar k_{in}}{2m} = \frac{m\hbar v_{in}}{m\hbar} = v_{in}$$

$$J_{reflected} = v_{ref} = 2V_{in} - v_{in}$$

J_{in} is in $\hat{x}$ direction $d\vec{a}$ is in $-\hat{x}$

$$\vec{J}_{in} \cdot d\vec{a} = -\frac{v_{in}}{\hbar} \text{ area}$$ probability flux flowing into surface

$$\vec{J}_{ref} \cdot d\vec{a} = \frac{(2V_{in} - v_{in})}{\hbar} \hat{x} \cdot (-\hat{x}) da = \frac{v_{in} - 2V_{in}}{\hbar} da$$

$$\oint \vec{J} \cdot d\vec{a} = -2V_{in} \text{ area}$$

more prob going in than out

$$\oint \vec{J} \cdot d\vec{a} = -\frac{\partial}{\partial t} \int \rho \, d\tau = -\frac{\partial}{\partial t} \text{ prob enclosed}$$

$d\tau$ = area dx

ρ = $\frac{\text{prob}}{\text{length}}$

prob enclosed increases as mirror moves to right

ρ is constant so it comes outside

$$= -\frac{\partial}{\partial t}(\rho_{in} + \rho_{ref}) \text{ area} \int_0^{x_f} dx$$

$\rho_{in} = 1$, $\rho_{ref} = 1$

$$= -2 \text{ area} \frac{\partial}{\partial t}(V_{in} t) = -2 \text{ area} V_{in}$$

So the net flux of probability into the surface is equal to the increase in probability within the surface as demanded by the differential form of conservation of probability.