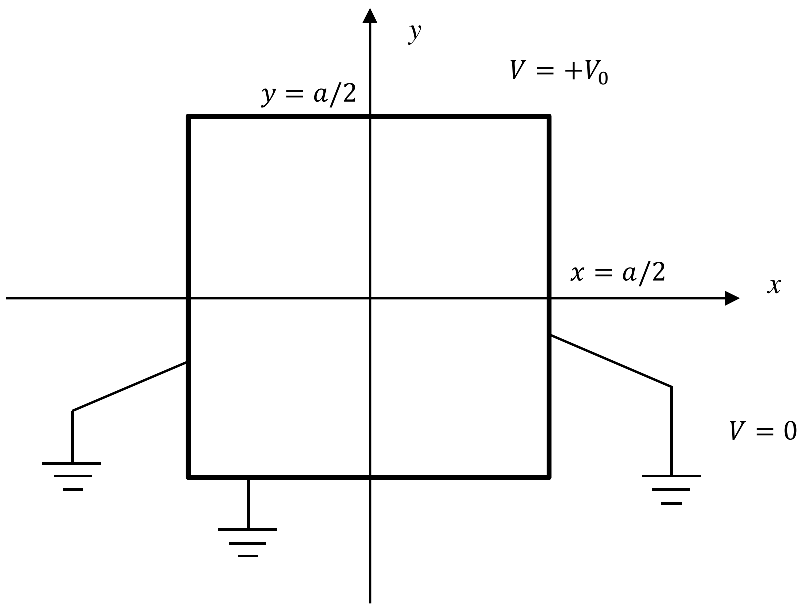
Phys 361 Homework 5

1) (based on Pollack and Stump 5.7)

Consider a long, rectangular, hollow metal pipe stretching out along the *z* axis. Let the sides be of length *a*, and let the cross-section of the pipe be centered on the origin. Hold the top side ($y = {}^{a}/_{2}$) at some voltage V_0, and ground the other three sides.



Show that the voltage everywhere inside the pipe is given by:

$$V(x,y) = \frac{4V_0}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\cos\left[(2n+1)\frac{\pi x}{a}\right]\frac{\sinh\left[(2n+1)\pi\left(\frac{y}{a}+\frac{1}{2}\right)\right]}{\sinh[(2n+1)\pi]}$$

And by "show that" I mean work forwards. Don't just plug the given V(x,y) into Laplace's equation and verify that it's a solution. Start from the beginning: Write a separable solution and the boundary conditions and proceed until you get to V.

Hint: Depending on the order in which you do things, you may or may not get an answer that looks like the above. If you don't, it might not be wrong – you might just need some hyperbolic trig identities to make it match. Possibly identities involving angle addition or a double angle formula.

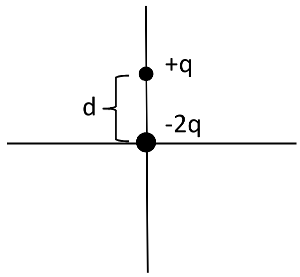
Additional hint: When you go to find your series coefficients by way of Fourier's trick, you might find that you need to use two conditions to find two coefficients. I personally found that exploiting the even/oddness of sinh and cosh made the coefficient solving easier.

2) Time for some separation of variables in spherical coordinates. Consider a hollow spherical shell of radius R. The top half of the shell ($0 \leq \theta \leq \frac{\pi}{2}$) has no charge on it. The bottom half of the shell ($\frac{\pi}{2} < \theta \leq \pi$) has some constant surface charge density σ_0. Find the potential both inside and outside of the shell.

Be sure to simplify your answer as much as possible. You will need a handbook, or the internets, or a mathphys book, or whatever, to find things like (but not limited to) the antiderivative of a Legendre polynomial. This is largely an exercise in manipulation, but a useful one: It demonstrates that special functions (like the Legendre polynomials) aren't really all that special, just unfamiliar. They still have properties and identities, just like our old friends like sine and cosine and exponentials and logs.

General note for the future: When you see some new special function, whether it be a Hermite polynomial, a Laguerre polynomial, or a Bessel function, a common first reaction is to curl up into a ball and cry. Don't do that. Instead, remind yourself that it's just another version of sine or cosine, governed by different-but-similar rules and identities.

3) Let's revisit multipole expansions. The serious business (i.e. grad school) way of handling multipole expansions is to write $V(\vec{x})$ as an expansion in spherical harmonics, which include Legendre polynomials. Take a look at the following charge distribution:



This is a point charge -2q on the origin with a point charge +q a distance d above.

Write the potential at large r (r being the distance from the origin to the point of observation) as a multipole expansion. I suggest simply writing the exact potential due to these point charges at some arbitrary location in terms of r, d, and θ, and then expanding that potential out in a power series the way we've done before. Or take the expansion we've done before and adapt it appropriately. Or, use the general expression (Pollack and Stump 3.91 or Griffiths problem 3.45) and evaluate. Whatever gets the job done.

However you get the potential, keep terms in the expansion up to the quadrupole ($1/r^3$), and write your potential function in terms of Legendre polynomials in $\cos\theta$, indicated by $P_n(\cos\theta)$. This isn't intended to be too big of an undertaking – we're just discovering that we can write things we've written before in terms of Legendre polynomials

4) (based on Pollack and Stump 5.11)

If we stick an object in an applied electric field, it polarizes – the positive and negative charges in the object that normally sit on top of each other separate out, with some excess positive to one side and some excess negative to the other side. The polarizability α is the constant of proportionality that relates the acquired dipole moment $\vec{p}$ to the applied electric field $\vec{E}_{app}$:

$$\vec{p} = \alpha\vec{E}_{app}$$

Find the polarizability of a solid conducting sphere of radius a. I'd suggest the following approach: First, have a look back at the example we've already done that involved a conducting sphere sitting in a constant E-field. Then use the induced charge density from that example plus the fundamental definition of a dipole moment from the multipole expansion section to calculate the dipole moment of the polarized sphere. You shouldn't need anything special from the chapter on polarization.