MATH 348 - Advanced Engineering Mathematics February 13, 2008
Homework 5, Spring 2008 **Due**: February 20, 2008

FOURIER SERIES AND ODE'S - FREQUENCY RESPONSE FUNCTIONS - DISCRETE AND FAST FOURIER TRANSFORMS

1. Consider the ODE, which is commonly used to model forced simple harmonic oscillation,

$$y'' + 9y = f(t), \tag{1}$$
$$f(t) = |t|, \; -\pi \leq t < \pi, \; f(t+2\pi) = f(t). \tag{2}$$

Since the forcing function (2) is a periodic function we can study (1) by expressing $f(t)$ as a Fourier series. [1] [2]

(a) Determine the real-Fourier series representation of $f(t)$.

(b) The solution to the homogenous problem associated with (1) is $y_h(t) = A\cos(3t) + B\sin(3t), \quad A, B \in \mathbb{R}$. Knowing this, if you were to use the method of undetermined coefficients[3] then what would your choice for the particular solution, $y_p(t)$? DO NOT SOLVE FOR THE UNKNOWN CONSTANTS

(c) What is the particular solution associated with the third Fourier mode of the forcing function found in (a)?[4]

(d) What is the long term behavior of the solution to (1) subject to (2)? What if the ODE had the form $y'' + 4y = f(t)$?

2. Given,

$$my'' + cy' + ky = f(t). \tag{3}$$

(a) Show that the frequency response function is given by, $\hat{g}(\omega) = \dfrac{1}{k + ic\omega - m\omega^2}$.[5]

(b) Show that for $m = k = 1$, $|\hat{g}(\omega)| = \sqrt{\hat{g}(\omega)\overline{\hat{g}(\omega)}} = \dfrac{1}{\sqrt{(\omega^2-1)^2 + c^2\omega^2}}$.

(c) Plot $|\hat{g}(\omega)|$ for $c^2 = \{0.01, 0.1, 1\}$. Comment on the frequency response of the system for when c^2 is large and when c^2 is small.

3. Assume that you are given four sample values (measurements) represented by the sample vector $\mathbf{f} = [0, \; 1, \; 4, \; , 9]^{\mathrm{T}}$. Calculate the Fourier matrix, $\mathbf{F}_4$ and discrete Fourier transform, $\hat{\mathbf{f}}$, associated with this sample.

4. Compute the fast Fourier transform for the previous sample data.

5. Visit the wikipedia pages for the discrete[6], fast[7], and fractional Fourier transforms[8] as well as the page on spectrum analyzers[9] and respond to the following:

(a) Explain the purpose of the modifiers DISCRETE, FAST, and FRACTIONAL associated with these Fourier transforms.

(b) If audio signals are continuous in time then why should one study the discrete Fourier transform?

(c) How are digital spectrum analyzers related to the discrete Fourier transform? What is the data output of a digital spectrum analyzer?

(d) Explain the difference between the fractional and nonfractional Fourier transforms in terms of the relationship between the rectangular function and the sinc function.

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[1] The advantage of expressing $f(t)$ as a Fourier series is its validity for any time t. The alternative would have been to construct a solution over each interval $n\pi < t < (n+1)\pi$ and then piece together the final solution assuming that the solution and its first derivative is continuous at each $t = n\pi$.

[2] It is worth noting that this method can be used by structural engineers to study the effects of periodic forcing on buildings and bridges.

[3] This is also known as the method of the 'lucky guess' in your differential equations text.

[4] Each term in a Fourier series is called a mode. The first mode is sometimes called the fundamental mode. The rest of the modes, after this fundamental mode, are just referenced by number. The third Fourier mode would be the third term of Fourier summation

[5] Recall that the frequency response function $\hat{g}$ is the Fourier transform of the Green's function, g, given by, $mg'' + cg' + kg = \delta(t)$.

[6] Introductory paragraph - `http://en.wikipedia.org/wiki/Discrete_Fourier_transform`

[7] Introductory paragraph - `http://en.wikipedia.org/wiki/Fast_Fourier_transform`

[8] Introduction and physical meaning of FFT - `http://en.wikipedia.org/wiki/Fractional_Fourier_transform`

[9] Introductory paragraph - `http://en.wikipedia.org/wiki/Spectrum_analyzer`