Homework 5

Section 4.2 12; Section 4.3 10a, 10e; Section 4.4 8

4.2.12 (10 pts) Prove the following proposition:

For all sets A, B, and C that are subsets of some universal set, if $A \cap B = A \cap C$ and $A^c \cap B = A^c \cap C$, then $B = C$.

Proof. In order to show the set equality, we will show $B \subseteq C$ and $C \subseteq B$.

B ⊆ C: Let $b \in B$. Then either $b \in A$ or $b \notin A$

Case 1: $b \in A$ **-** Then $b \in A \cap B \Rightarrow b \in A \cap C \Rightarrow b \in C$. Thus, $B \subseteq C$.

Case 2: $b \notin A$ **-** Then $b \in A^c \Rightarrow b \in A^c \cap B \Rightarrow b \in A^c \cap C \Rightarrow b \in C$. Thus, $B \subseteq C$.

C ⊆ B: Let $c \in C$. Then either $c \in A$ or $c \notin A$

Case 1: $c \in A$ **-** Then $c \in A \cap C \Rightarrow c \in A \cap B \Rightarrow c \in B$. Thus, $C \subseteq B$.

Case 2: $c \notin A$ **-** Then $c \in A^c \Rightarrow c \in A^c \cap C \Rightarrow c \in A^c \cap B \Rightarrow c \in B$. Thus, $C \subseteq B$.

Therefore, we see in all cases $B \subseteq C$ and $C \subseteq B$. Thus $B = C$. □

4.3.10 (10 pts) Let A and B be subsets of some universal set U. Prove or disprove each of the following:

a. $A - (A \cap B^c) = (A \cap B)$

Proof. Using properties of set operations, we have

$$\begin{aligned} A - (A \cap B^c) &= A \cap (A \cap B^c)^c \\ &= A \cap (A^c \cup B) \\ &= (A \cap A^c) \cup (A \cap B) \\ &= \varnothing \cup (A \cap B) \\ &= A \cap B \end{aligned}$$

□

e. $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$

Proof. Using properties of set operations, we have

$$\begin{aligned} (A \cup B) - (A \cap B) &= (A \cup B) \cap (A \cap B)^c \\ &= (A \cup B) \cap (A^c \cup B^c) \\ &= ((A \cup B) \cap A^c) \cup ((A \cup B) \cap B^c) \\ &= ((A \cap A^c) \cup (B \cap A^c)) \cup ((A \cap B^c) \cup (B \cap B^c)) \\ &= (\varnothing \cup (B \cap A^c)) \cup ((A \cap B^c) \cup \varnothing) \\ &= (B \cap A^c) \cup (A \cap B^c) \\ &= (B - A) \cup (A - B) \\ &= (A - B) \cup (B - A) \end{aligned}$$

□

4.4.8 (10 pts) Let A and B be nonempty sets. Prove that $A \times B = B \times A$ if and only if $A = B$.

Proof.

⇒: We will assume $A \times B = B \times A$ and show that $A = B$.
Let $a \in A$. Then for all $b \in B$,

$$(a, b) \in A \times B \Rightarrow (a, b) \in B \times A \Rightarrow a \in B.$$

Thus $A \subseteq B$.
Similarly, for $b \in B$, for all $a \in A$,

$$(b, a) \in B \times A \Rightarrow (b, a) \in A \times B \Rightarrow b \in A.$$

Thus, $B \subseteq A$ and we see that $A = B$.

⇐: We will assume $A = B$ and show that $A \times B = B \times A$.
Let $(a, b) \in A \times B$. Then for $a \in A \Rightarrow a \in B$ and for $b \in B \Rightarrow b \in A$.
Thus, $(a, b) \in B \times A \Rightarrow A \times B \subseteq B \times A$.
Similarly, for $(b, a) \in B \times A$, we also have $(b, a) \in A \times B$.
Thus, $B \times A \subseteq A \times B$ and therefore $A \times B = B \times A$.

Note: This proof, if worded carefully, can be shortened considerably.

⇒: Let $(a, b) \in A \times B \Leftrightarrow a \in A$ and $b \in B$.
Since $A \times B = B \times A$, $(a, b) \in B \times A \Rightarrow a \in B$ and $b \in A$. Thus, $A = B$.

⇐: Since $A = B$, we have for all $a \in A$ and all $b \in B$, $a \in B$ and $b \in A$.
Thus for all $(a, b) \in A \times B, (a, b) \in B \times A$ and for all $(b, a) \in B \times A$, $(b, a) \in A \times B$.
Thus, $A \times B = B \times A$.

□