Dispersion and Ultrashort Pulses

Angular dispersion and group-velocity dispersion

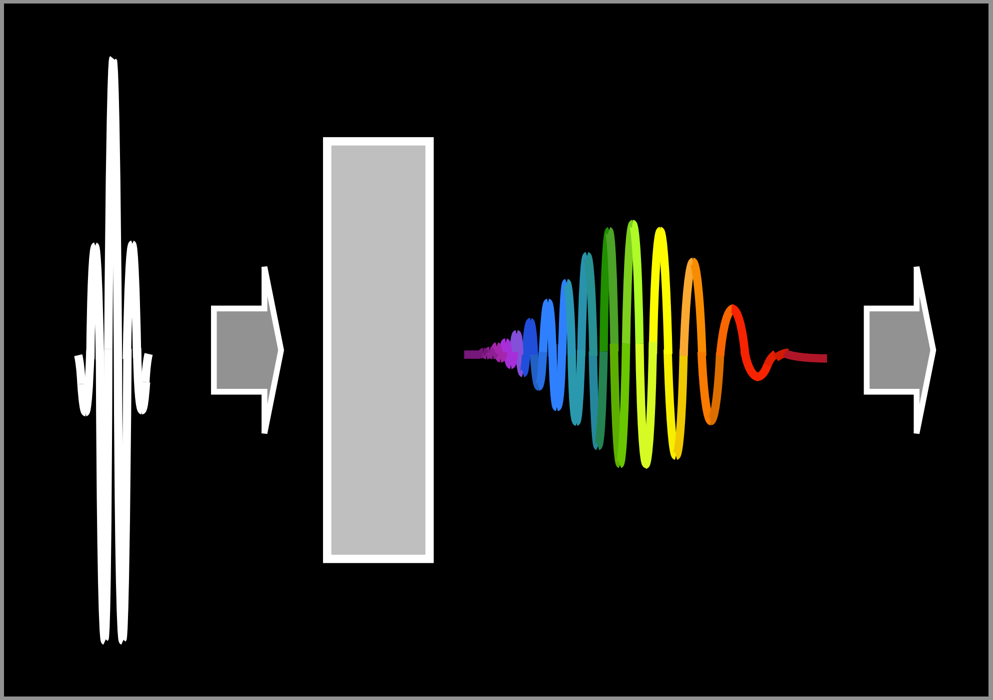
Phase and group velocities

Group-delay dispersion

Negative group-delay dispersion

Pulse compression

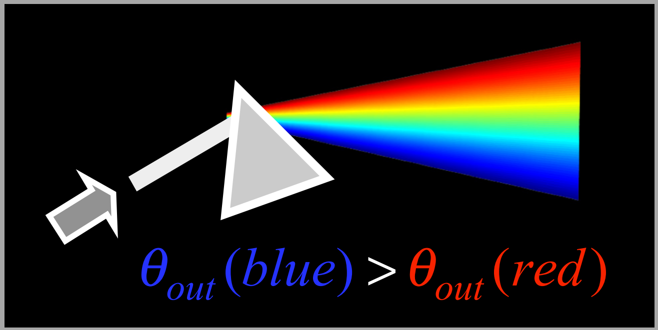
Chirped mirrors



Dispersion in Optics

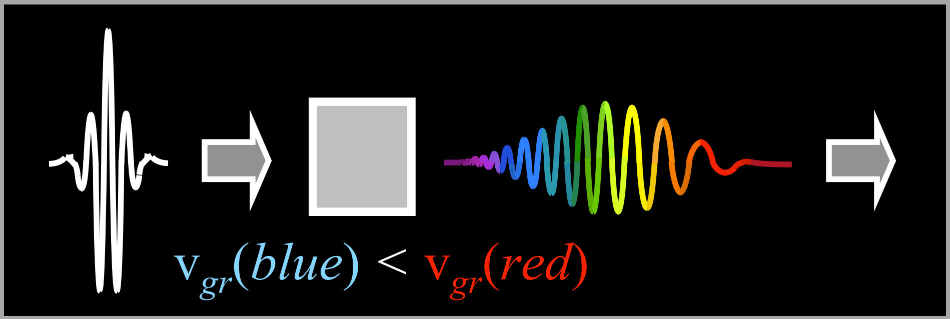
The dependence of the refractive index on wavelength has two effects on a pulse, one in space and the other in time.

Dispersion disperses a pulse in space (angle):



"Angular dispersion"
$dn/d\lambda$

Dispersion also disperses a pulse in time:



"Chirp"
$d^2n/d\lambda^2$

Both of these effects play major roles in ultrafast optics.

Calculating the group velocity

$$v_g \equiv d\omega / dk$$

Now, ω is the same in or out of the medium, but $k = k_0 n$, where k_0 is the k-vector in vacuum, and n is what depends on the medium.
So it's easier to think of ω as the independent variable:

$$\boxed{v_g \equiv \left[dk / d\omega\right]^{-1}}$$

Using $k = \omega\, n(\omega) / c_0$, calculate: $dk / d\omega = (n + \omega\, dn/d\omega) / c_0$

$$v_g = c_0 / (n + \omega\, dn/d\omega) = (c_0 / n) / (1 + \omega/n\; dn/d\omega)$$

Finally:

$$\boxed{v_g = v_{phase} / \left(1 + \frac{\omega}{n}\frac{dn}{d\omega}\right)}$$

So the group velocity equals the phase velocity when $dn/d\omega = 0$, such as in vacuum. Otherwise, since n usually increases with ω, $dn/d\omega > 0$, and:

$$v_g < v_{phase}.$$

Calculating group velocity vs. wavelength

We more often think of the refractive index in terms of wavelength, so let's write the group velocity in terms of the vacuum wavelength λ_0.

Use the chain rule: $$\frac{dn}{d\omega} = \frac{dn}{d\lambda_0}\frac{d\lambda_0}{d\omega}$$

Now, $\lambda_0 = 2\pi c_0/\omega$, so: $$\frac{d\lambda_0}{d\omega} = \frac{-2\pi c_0}{\omega^2} = \frac{-2\pi c_0}{(2\pi c_0/\lambda_0)^2} = \frac{-\lambda_0^2}{2\pi c_0}$$

Recalling that: $$\mathrm{v}_g = \left(\frac{c_0}{n}\right) \Big/ \left[1 + \frac{\omega}{n}\frac{dn}{d\omega}\right]$$

we have: $$\mathrm{v}_g = \left(\frac{c_0}{n}\right) \Big/ \left[1 + \frac{\cancel{2\pi c_0}}{n\lambda_0}\left\{\frac{dn}{d\lambda_0}\left(\frac{-\lambda_0^2}{\cancel{2\pi c_0}}\right)\right\}\right]$$

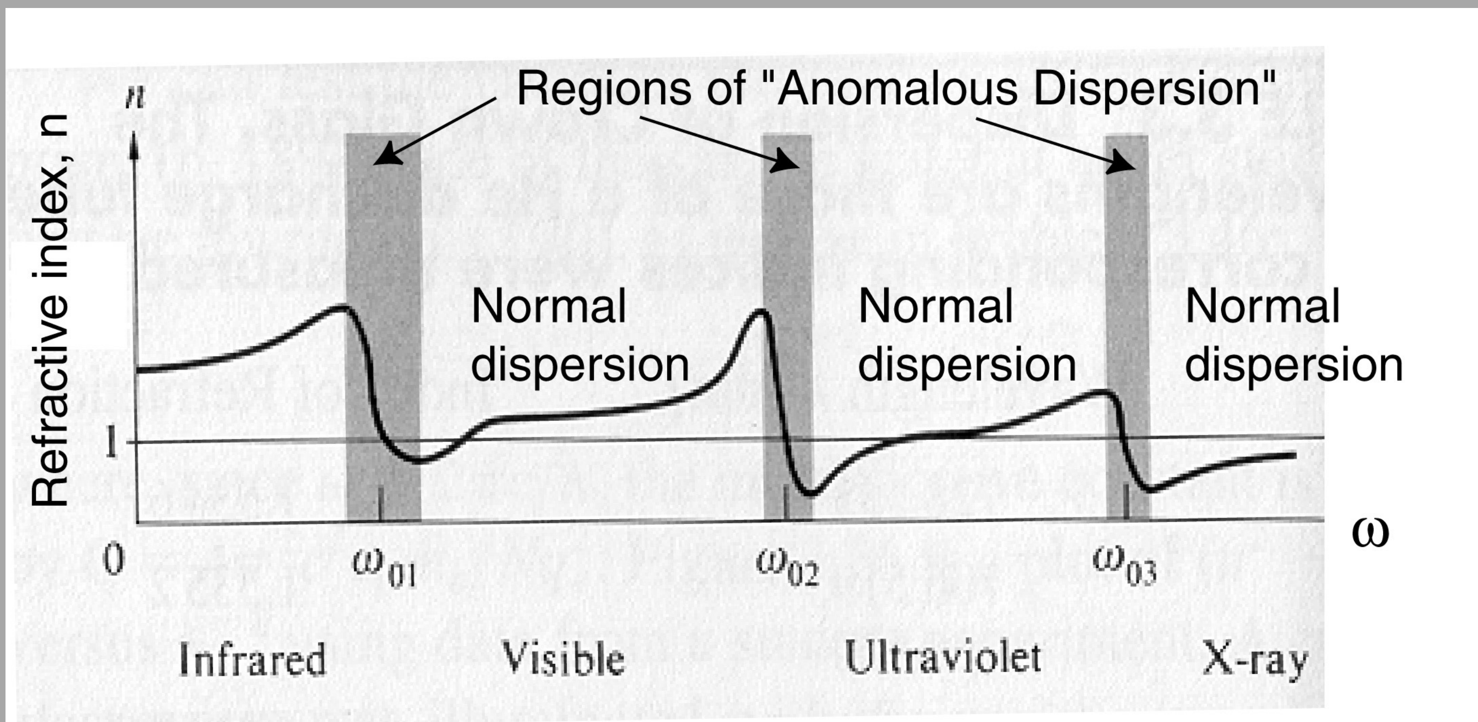
or:

$$\boxed{\mathrm{v}_g = \left(\frac{c_0}{n}\right) \Big/ \left(1 - \frac{\lambda_0}{n}\frac{dn}{d\lambda_0}\right)}$$

The group velocity is less than the phase velocity in non-absorbing regions.

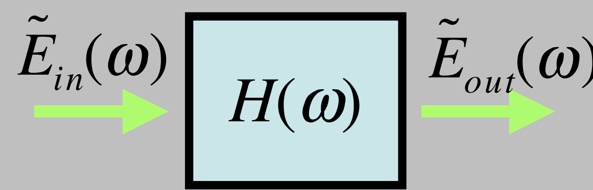
$$v_g = c_0 / (n + \omega\, dn/d\omega)$$

Except in regions of anomalous dispersion (which are absorbing), $dn/d\omega$ is positive, that is, near a resonance. So $v_g < v_{phase}$ for these frequencies!



Spectral Phase and Optical Devices

Recall that the effect of a linear passive optical device (i.e., lenses, prisms, etc.) on a pulse is to multiply the frequency-domain field by a transfer function:



Optical device or medium

$$\tilde{E}_{out}(\omega) \;=\; H(\omega)\,\tilde{E}_{in}(\omega)$$

where $H(\omega)$ is the transfer function of the device/medium:

$\exp[-\alpha(\omega)L/2]$ for a medium

$$H(\omega) \;=\; B_H(\omega)\,\exp[-i\varphi_H(\omega)]$$

Since we also write $E(\omega) = \sqrt{S(\omega)}\,\exp[-i\varphi(\omega)]$, the spectral phase of the output light will be:

$$\boxed{\varphi_{out}(\omega) \;=\; \varphi_H(\omega) + \varphi_{in}(\omega)}$$

We simply add spectral phases.

Note that we CANNOT add the temporal phases!

$$\phi_{out}(t) \;\neq\; \phi_H(t) + \phi_{in}(t)$$

The Group-Velocity Dispersion (GVD)

The phase due to a medium is: $\varphi_H(\omega) = n(\omega)\, k\, L = k(\omega)\, L$

To account for dispersion, expand the phase (k-vector) in a Taylor series:

$$k(\omega)L \;=\; k(\omega_0)L \;+\; k'(\omega_0)\left[\omega-\omega_0\right]L \;+\; \tfrac{1}{2}k''(\omega_0)\left[\omega-\omega_0\right]^2 L + \ldots$$

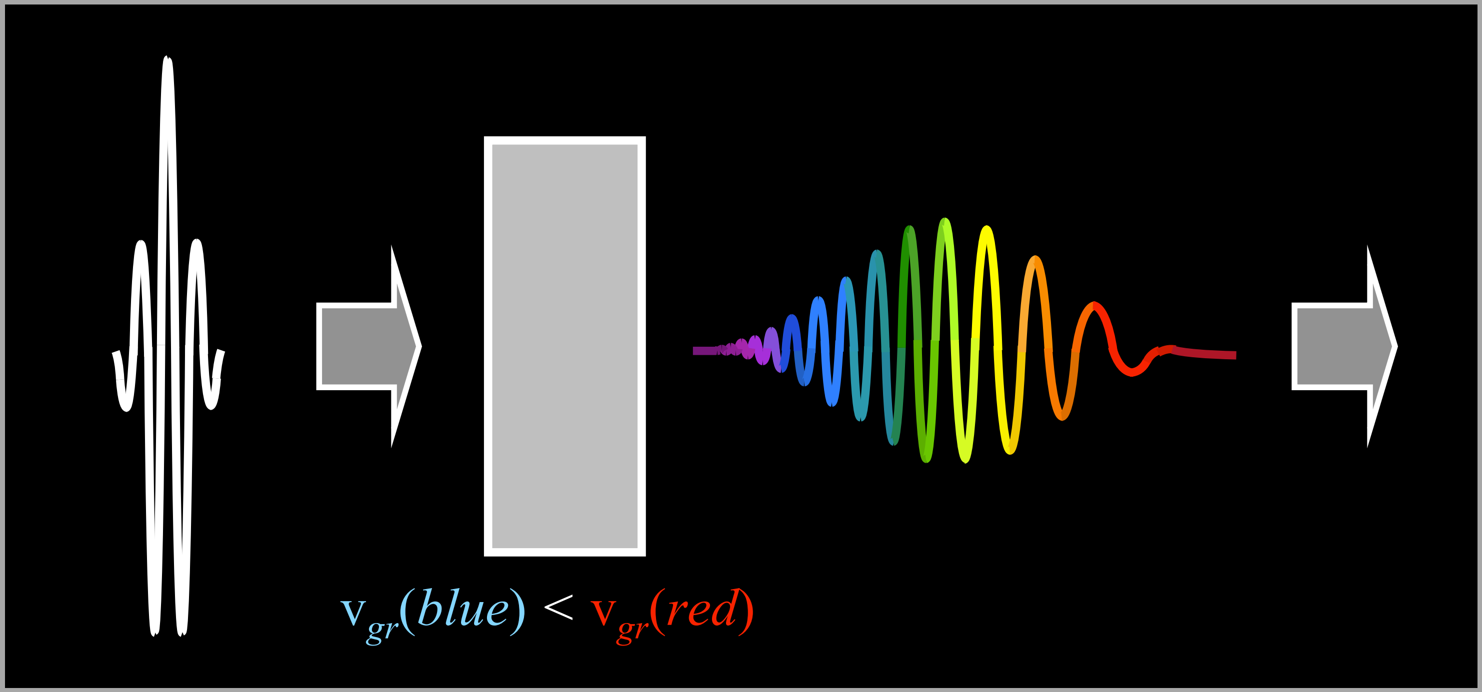
$$k(\omega_0) = \frac{\omega_0}{\mathrm{v}_\phi(\omega_0)} \qquad k'(\omega_0) = \frac{1}{\mathrm{v}_g(\omega_0)} \qquad k''(\omega) = \frac{d}{d\omega}\left[\frac{1}{\mathrm{v}_g}\right]$$

The first few terms are all related to important quantities.
The third one is new: the variation in group velocity with frequency:

$$\boxed{k''(\omega) = \frac{d}{d\omega}\left[\frac{1}{\mathrm{v}_g}\right]}$$ is the "**group velocity dispersion**."

The effect of group velocity dispersion

GVD means that the group velocity will be different for different wavelengths in the pulse.



Because ultrashort pulses have such large bandwidths, GVD is a bigger issue than for cw light.

Calculation of the GVD (in terms of wavelength)

Recall that:

$$\frac{d\lambda_0}{d\omega} = \frac{-\lambda_0^2}{2\pi c_0} \qquad \frac{d}{d\omega} = \frac{d\lambda_0}{d\omega}\frac{d}{d\lambda_0} = \frac{-\lambda_0^2}{2\pi c_0}\frac{d}{d\lambda_0}$$

and

$$\mathrm{v}_g = c \Big/ \left(n - \lambda_0 \frac{dn}{d\lambda_0} \right)$$

Okay, the GVD is:

$$\frac{d}{d\omega}\left[\frac{1}{\mathrm{v}_g}\right] = \frac{-\lambda_0^2}{2\pi c_0}\frac{d}{d\lambda_0}\left[\frac{1}{c}\left(n - \lambda_0\frac{dn}{d\lambda_0}\right)\right] = \frac{-\lambda_0^2}{2\pi c_0^2}\frac{d}{d\lambda_0}\left[n - \lambda_0\frac{dn}{d\lambda_0}\right]$$

$$= \frac{-\lambda_0^2}{2\pi c_0^2}\left[\frac{dn}{d\lambda_0} - \lambda_0\frac{d^2 n}{d\lambda_0^2} - \frac{dn}{d\lambda_0}\right]$$

Simplifying:

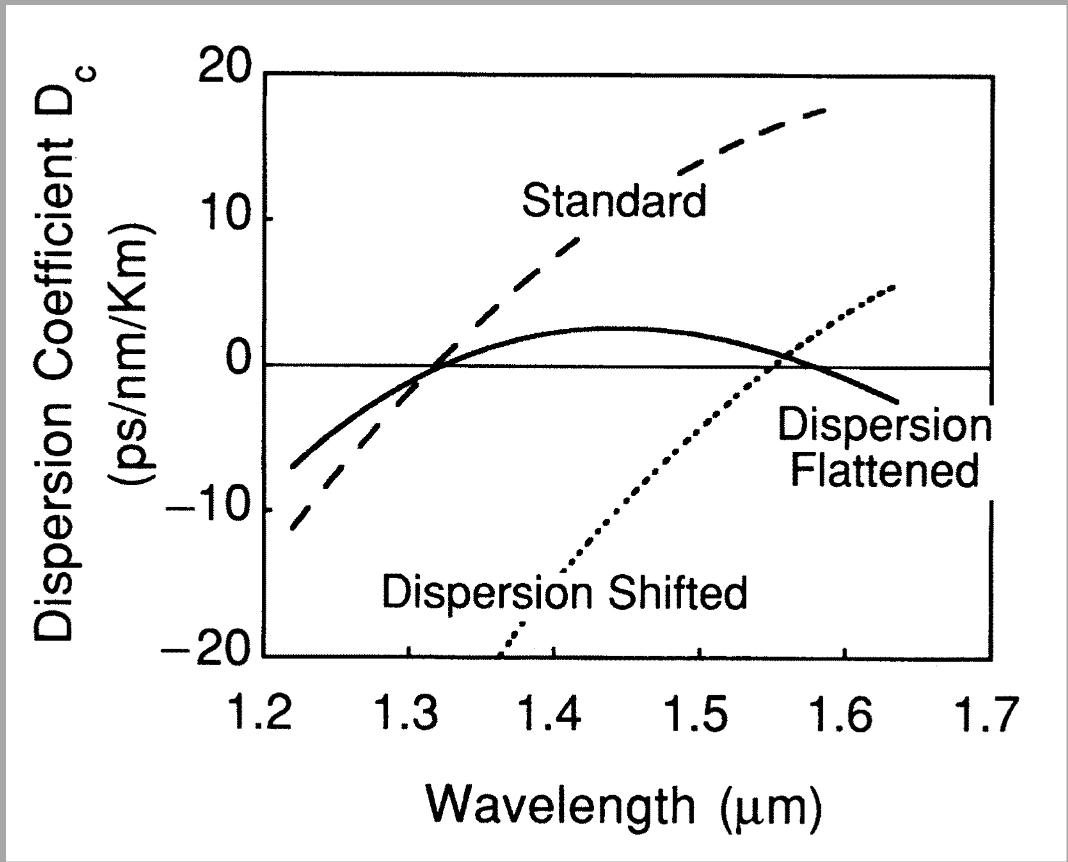
$$\boxed{GVD \equiv k''(\omega_0) = \frac{\lambda_0^3}{2\pi c_0^2}\frac{d^2 n}{d\lambda_0^2}}$$

Units:
ps^2/km or
(s/m)/Hz or
s/Hz/m

Dispersion parameters for various materials

material	λ [nm]	$n(\lambda)$	$\frac{dn}{d\lambda}\cdot 10^{-2}\left[\frac{1}{\mu m}\right]$	$\frac{d^2n}{d\lambda^2}\cdot 10^{-1}\left[\frac{1}{\mu m^2}\right]$	$\frac{dn^3}{d\lambda^3}\left[\frac{1}{\mu m^3}\right]$	$T_g\left[\frac{fs}{mm}\right]$	$GDD\left[\frac{fs^2}{mm}\right]$	$TOD\left[\frac{fs^3}{mm}\right]$
BK7	400	1,5308	-13,17	10,66	-12,21	5282	120,79	40,57
	500	1,5214	-6,58	3,92	-3,46	5185	86,87	32,34
	600	1,5163	-3,91	1,77	-1,29	5136	67,52	29,70
	800	1,5108	-1,97	0,48	-0,29	5092	43,96	31,90
	1000	1,5075	-1,40	0,15	-0,09	5075	26,93	42,88
	1200	1,5049	-1,23	0,03	-0,04	5069	10,43	66,12
SF10	400	1,7783	-52,02	59,44	-101,56	6626	673,68	548,50
	500	1,7432	-20,89	15,55	-16,81	6163	344,19	219,81
	600	1,7267	-11,00	6,12	-4,98	5980	233,91	140,82
	800	1,7112	-4,55	1,58	-0,91	5830	143,38	97,26
	1000	1,7038	-2,62	0,56	-0,27	5771	99,42	92,79
	1200	1,6992	-1,88	0,22	-0,10	5743	68,59	107,51
Sapphire	400	1,7866	-17,20	13,55	-15,05	6189	153,62	47,03
	500	1,7743	-8,72	5,10	-4,42	6064	112,98	39,98
	600	1,7676	-5,23	2,32	-1,68	6001	88,65	37,97
	800	1,7602	-2,68	0,64	-0,38	5943	58,00	42,19
	1000	1,7557	-1,92	0,20	-0,12	5921	35,33	57,22
	1200	1,7522	-1,70	0,04	-0,05	5913	13,40	87,30
Quartz	300	1,4878	-30,04	34,31	-54,66	5263	164,06	46,49
	400	1,4701	-11,70	9,20	-10,17	5060	104,31	31,49
	500	1,4623	-5,93	3,48	-3,00	4977	77,04	26,88
	600	1,4580	-3,55	1,59	-1,14	4934	60,66	25,59
	800	1,4533	-1,80	0,44	-0,26	4896	40,00	28,43
	1000	1,4504	-1,27	0,14	-0,08	4880	24,71	38,73
	1200	1,4481	-1,12	0,03	-0,03	4875	9,76	60,05

GVD in optical fibers



Note that fiber folks define GVD as the negative of ours.

Sophisticated cladding structures, i.e., index profiles have been designed and optimized to produce a waveguide dispersion that modifies the bulk material dispersion

GVD yields group delay dispersion (GDD).

We can define delays in terms of the velocities and the medium length L.

The phase delay:

$$k(\omega_0) = \frac{\omega_0}{\mathrm{v}_\phi(\omega_0)} \qquad \text{so:} \qquad t_\phi = \frac{L}{\mathrm{v}_\phi(\omega_0)} = \frac{k(\omega_0)L}{\omega_0}$$

The group delay:

$$k'(\omega_0) = \frac{1}{\mathrm{v}_g(\omega_0)} \qquad \text{so:} \qquad t_g(\omega_0) = \frac{L}{\mathrm{v}_g(\omega_0)} = k'(\omega_0)L$$

The group delay dispersion (GDD):

$GDD = GVD\ L$

$$k''(\omega) = \frac{d}{d\omega}\left[\frac{1}{\mathrm{v}_g}\right] \qquad \text{so:} \qquad \boxed{GDD = \frac{d}{d\omega}\left[\frac{1}{\mathrm{v}_g}\right]L = k''(\omega)L}$$

Units: fs^2 or fs/Hz

Manipulating the phase of light

Recall that we expand the spectral phase of the pulse in a Taylor Series:

$$\varphi(\omega) \quad = \quad \varphi_0 + \varphi_1 \, [\omega - \omega_0] + \varphi_2 \, [\omega - \omega_0]^2 / 2! + \ldots$$

and we do the same for the spectral phase of the optical medium, H:

$$\varphi_H(\omega) \quad = \quad \underbrace{\varphi_{H0}}_{\text{phase}} + \underbrace{\varphi_{H1}}_{\text{group delay}} \, [\omega - \omega_0] + \underbrace{\varphi_{H2}}_{\text{group delay dispersion (GDD)}} \, [\omega - \omega_0]^2 / 2! + \ldots$$

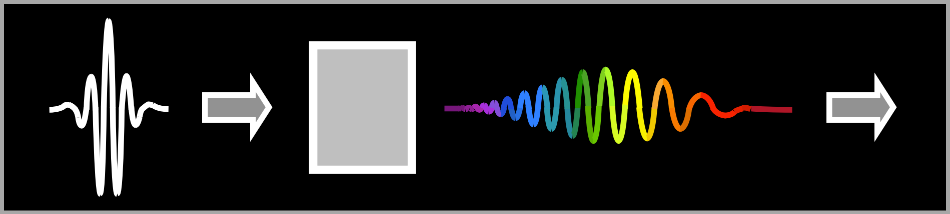
So, to manipulate light, we must add or subtract **spectral-phase terms**.

For example, to eliminate the linear chirp (second-order spectral phase), we must design an optical device whose second-order spectral phase cancels that of the pulse:

$$\varphi_2 + \varphi_{H2} = 0 \qquad \text{i.e.,} \qquad \left.\frac{d^2\varphi}{d\omega^2}\right|_{\omega_0} + \left.\frac{d^2\varphi_H}{d\omega^2}\right|_{\omega_0} = 0$$

Propagation of the pulse manipulates it.

Dispersive pulse broadening is unavoidable.



If φ_2 is the pulse 2nd-order spectral phase on entering a medium, and $k''L$ is the 2nd-order spectral phase of the medium, then the resulting pulse 2nd-order phase will be the sum: $\varphi_2 + k''L$.

A linearly chirped input pulse has 2nd-order phase: $\varphi_{2,in} = \dfrac{\beta/2}{\alpha^2+\beta^2}$

(This result pulls out the ½ in the Taylor Series.)

Emerging from a medium, its 2nd-order phase will be:

$$\varphi_{2,out} = \frac{\beta/2}{\alpha^2+\beta^2} + GDD = \frac{\beta/2}{\alpha^2+\beta^2} + \frac{\lambda_0^3}{2\pi c_0^2}\frac{d^2 n}{d\lambda_0^2} L \longleftarrow$$

This result, with the spectrum, can be inverse Fourier-transformed to yield the pulse.

A positively chirped pulse will broaden further;
a negatively chirped pulse will shorten.
Too bad material GDD is always positive in the visible and near-IR…

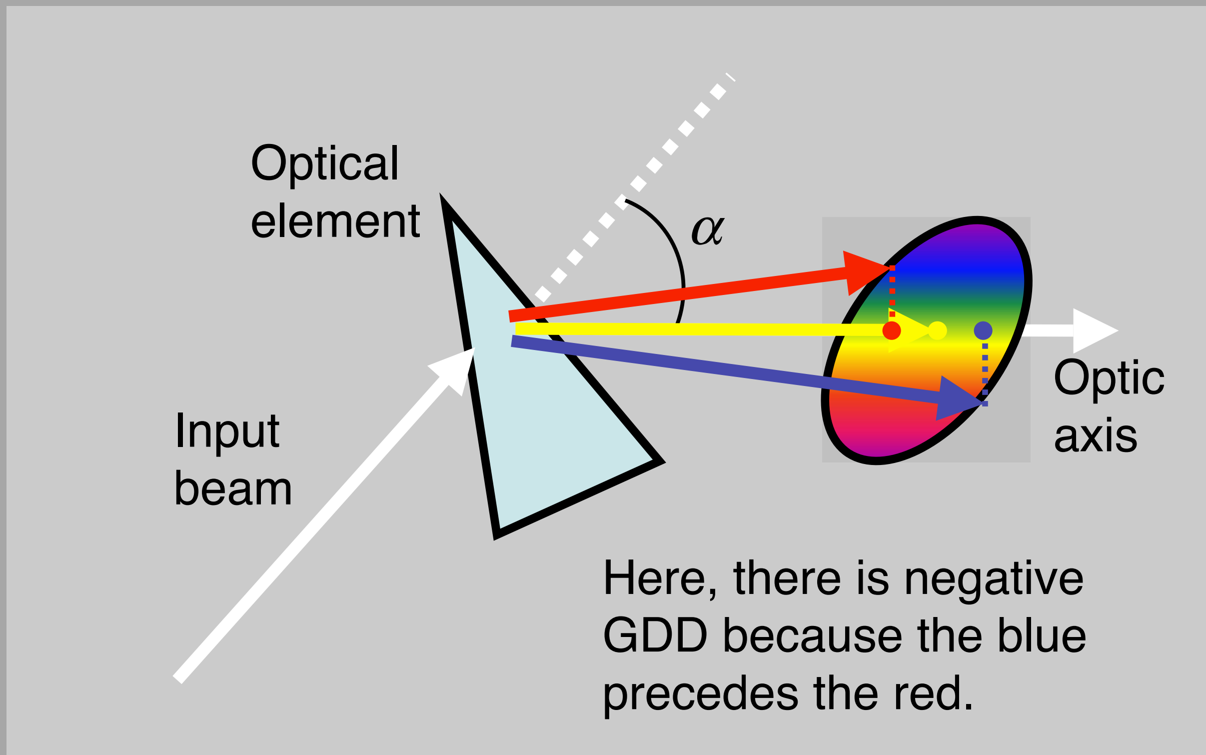
So how can we generate negative GDD?

This is a big issue because pulses spread further and further as they propagate through materials.

We need a way of generating negative GDD to compensate.

Angular dispersion yields negative GDD.

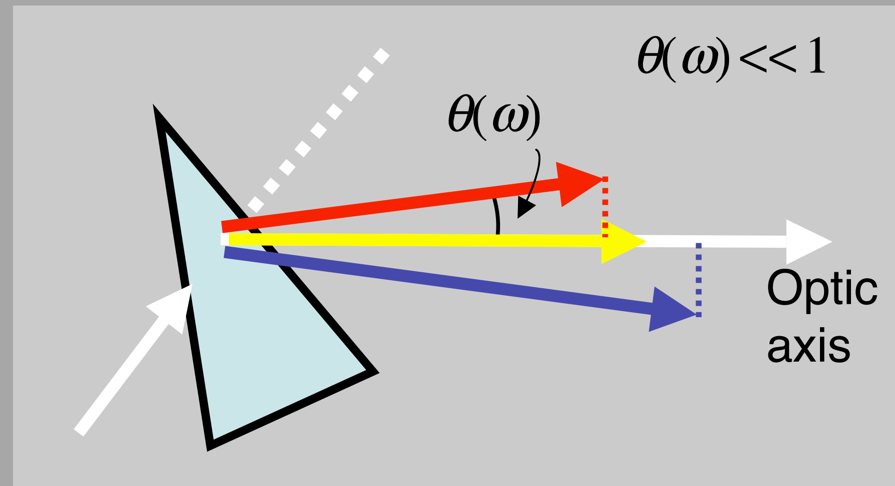
Suppose that an optical element introduces angular dispersion.



We'll need to compute the projection onto the optic axis (the propagation direction of the center frequency of the pulse).

Negative GDD

Taking the projection of $\vec{k}(\omega)$ onto the optic axis, a given frequency ω sees a phase delay of $\varphi(\omega)$:



We're considering only the GDD due to dispersion and not that of the prism itself. So $n = 1$ (that of the air after the prism).

$$\varphi(\omega) = \vec{k}(\omega) \cdot \overset{z}{\vec{r}_{optic\ axis}}$$

$$= k(\omega)\, z \cos[\theta(\omega)]$$

$$= (\omega / c)\, z \cos[\theta(\omega)]$$

$$d\varphi / d\omega = (z / c) \cos(\theta) - (\omega / c)\, z \sin(\theta)\; d\theta / d\omega$$

$$\frac{d^2\varphi}{d\omega^2} = -\cancel{\frac{z}{c}\sin(\theta)\frac{d\theta}{d\omega}} - \cancel{\frac{z}{c}\sin(\theta)\frac{d\theta}{d\omega}} - \omega\frac{z}{c}\cos(\theta)\left(\frac{d\theta}{d\omega}\right)^2 - \omega\cancel{\frac{z}{c}\sin(\theta)\frac{d^2\theta}{d\omega^2}}$$

But $\theta << 1$, so the sine terms can be neglected, and $\cos(\theta) \sim 1$.

Angular dispersion yields negative GDD.

$$\Rightarrow \quad \boxed{\left.\frac{d^2\varphi}{d\omega^2}\right|_{\omega_0} \approx -\frac{\omega_0 z}{c}\left(\left.\frac{d\theta}{d\omega}\right|_{\omega_0}\right)^2}$$

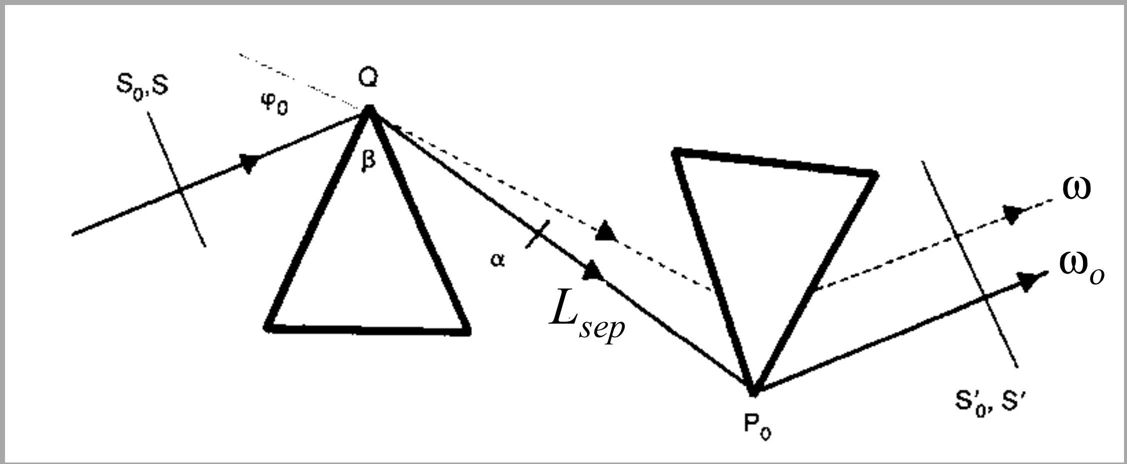
The GDD due to angular dispersion is always ***negative***!

Also, note that it doesn't matter where the angular dispersion came from (whether a prism or a grating).

And the negative GDD due to prism dispersion is usually much greater than that from the material of the prism.

A prism pair has negative GDD.

How can we use dispersion to introduce negative chirp conveniently?



Assume Brewster angle incidence and exit angles.

$$\left.\frac{d^2\varphi}{d\omega^2}\right|_{\omega_0} \approx -4L_{sep}\frac{\lambda_0^3}{2\pi c^2}\left(\left.\frac{dn}{d\lambda}\right|_{\lambda_0}\right)^2 + L_{prism}\frac{\lambda_0^3}{2\pi c^2}\left.\frac{d^2n}{d\lambda^2}\right|_{\lambda_0}$$

Always positive (in visible and near-IR)

Always negative!

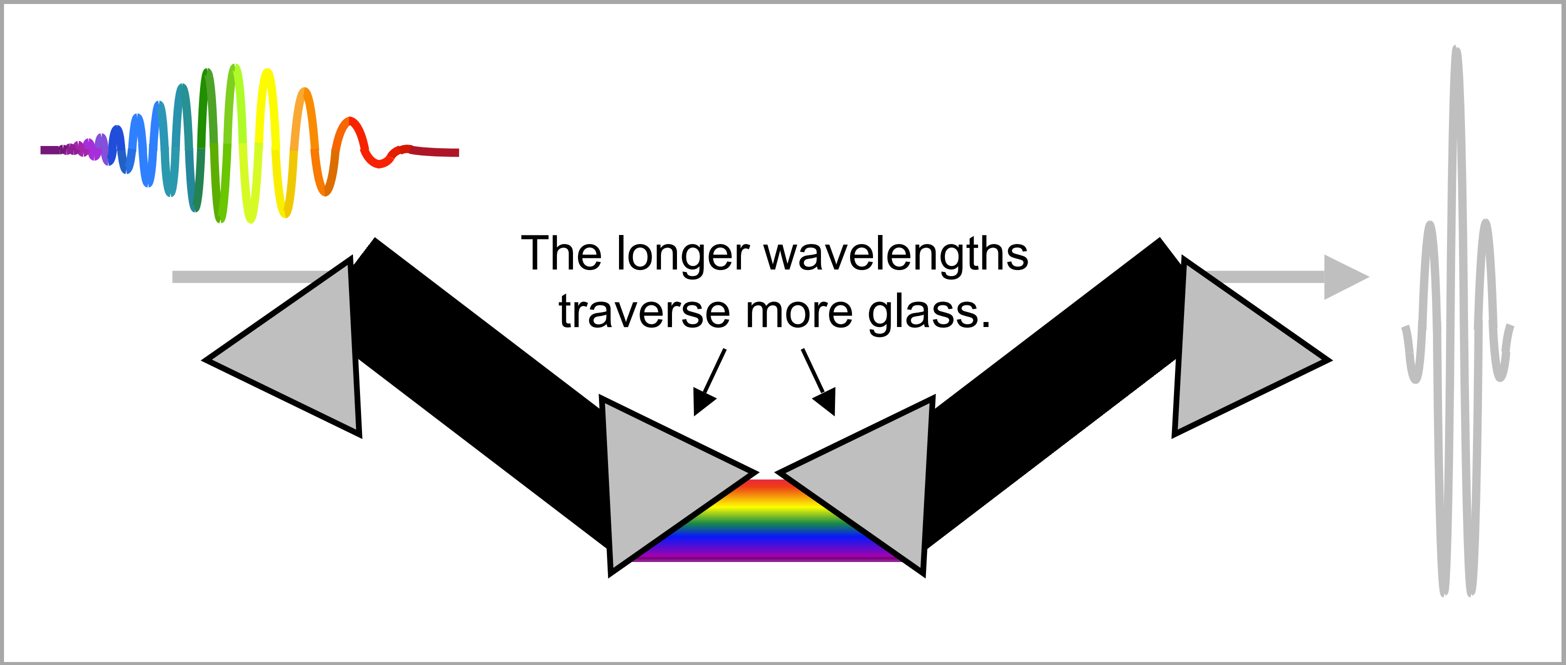
This term assumes that the beam grazes the tip of each prism

This term allows the beam to pass through an additional length, L_{prism}, of prism material.

Vary L_{sep} or L_{prism} to tune the GDD!

Pulse Compressor

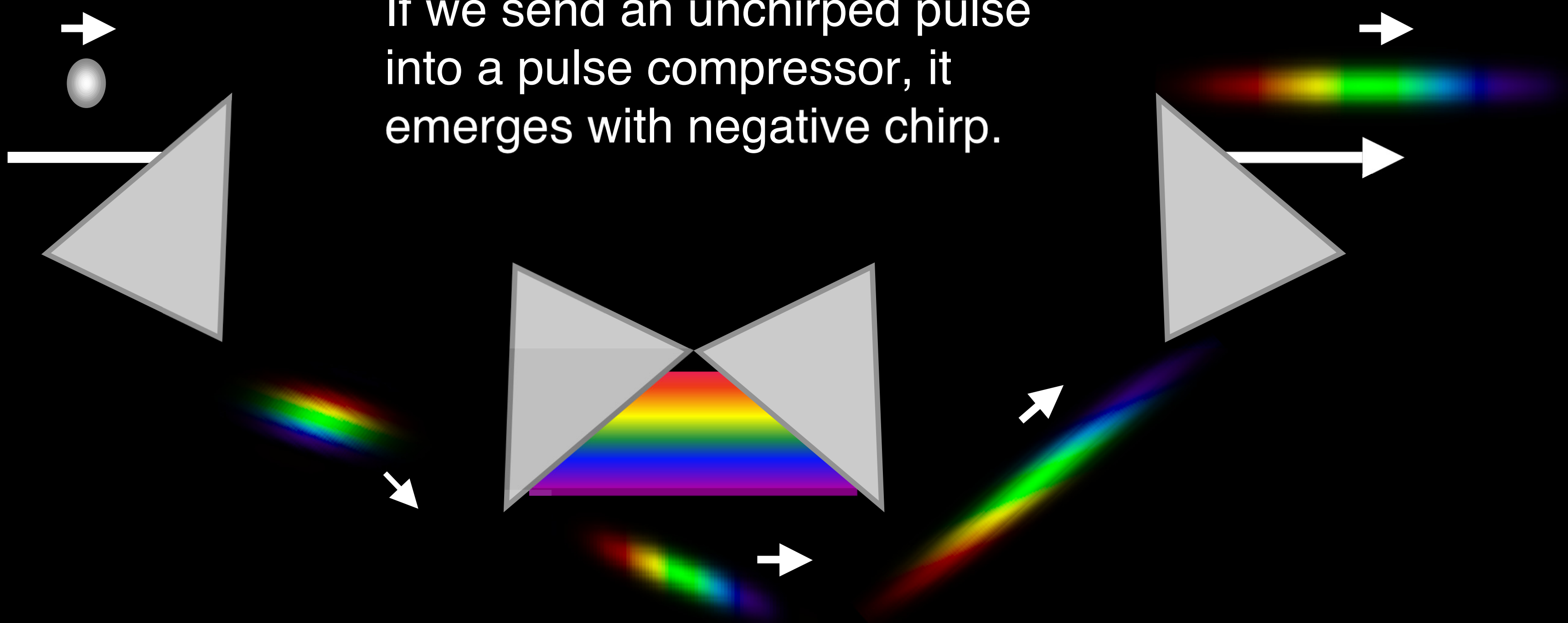
This device has **negative** group-delay dispersion and hence can compensate for propagation through materials (i.e., for positive chirp).



It's routine to stretch and then compress ultrashort pulses by factors of >1000.

What does the pulse look like inside a pulse compressor?

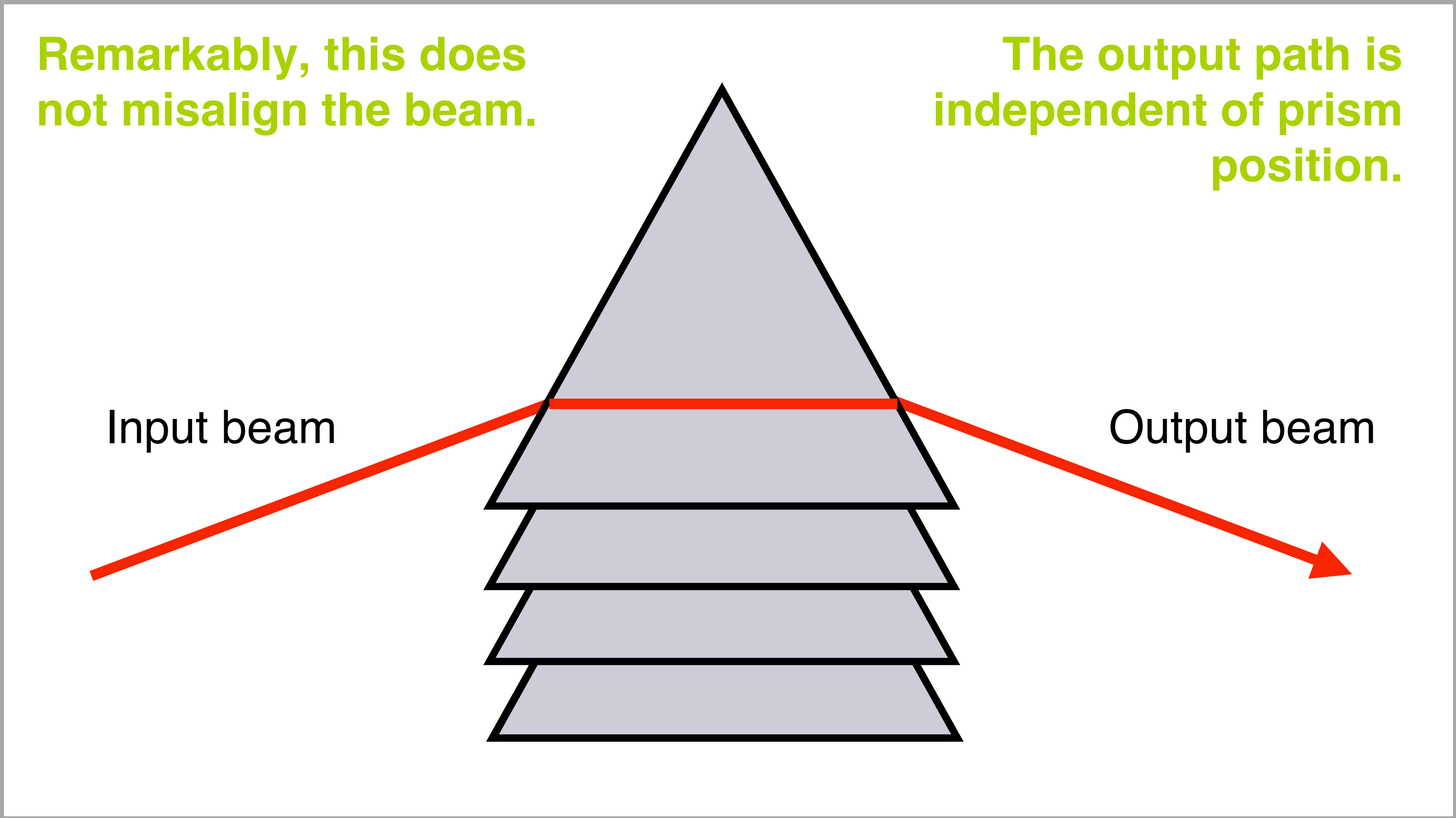
If we send an unchirped pulse into a pulse compressor, it emerges with negative chirp.



Note the unintuitive color variation of the pulse after the first prism.
To see the effect on a positively chirped pulse, read right to left!

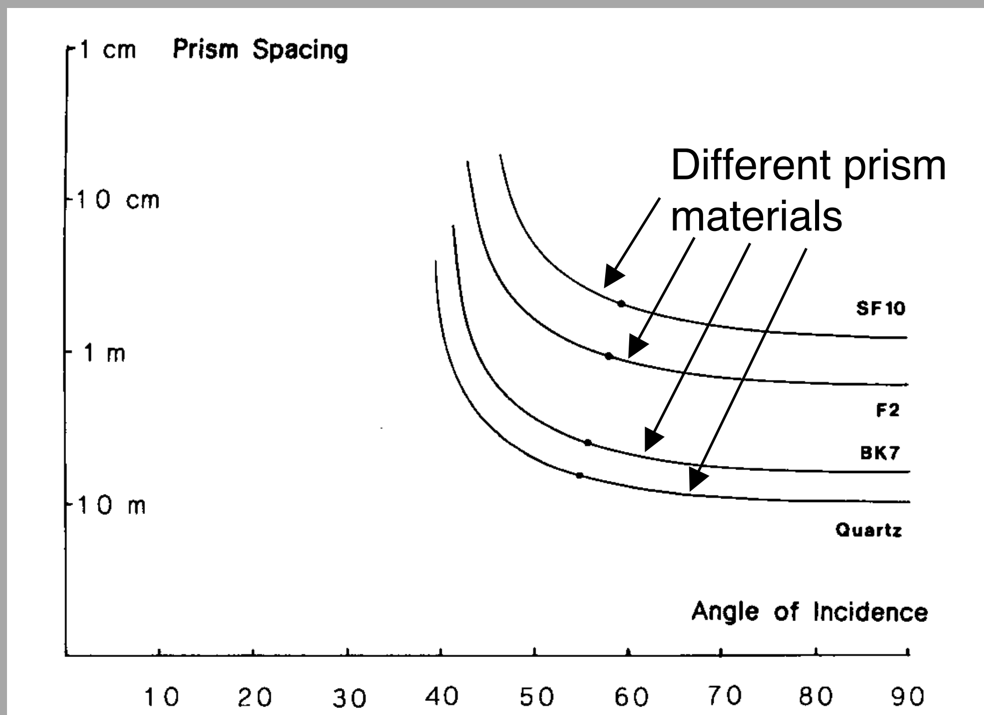
Adjusting the GDD maintains alignment.

Any prism in the compressor can be translated perpendicular to the beam path to add glass and reduce the magnitude of negative GDD.



The required separation between prisms in a pulse compressor can be large.

The GDD $\propto$ the prism separation and the square of the dispersion.



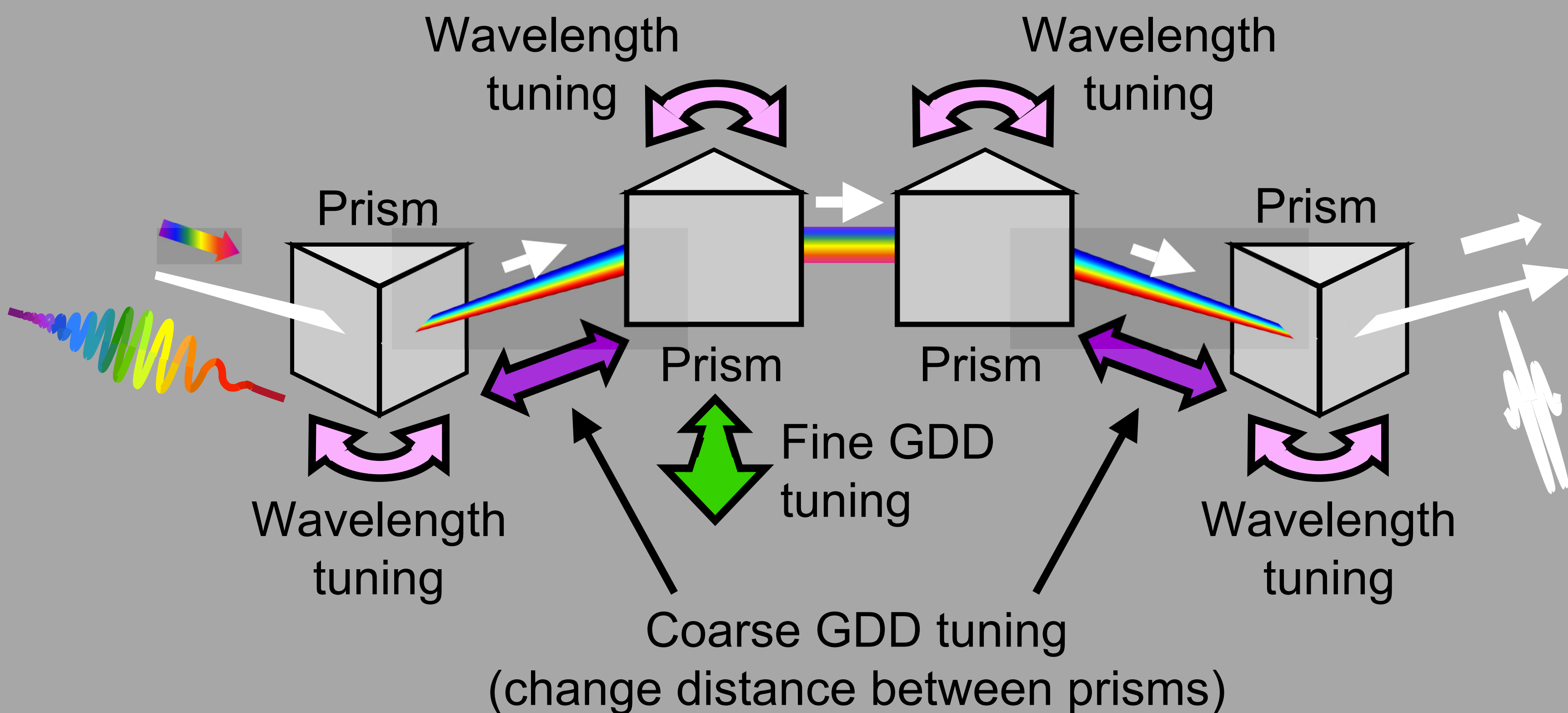
Compression of a 1-ps, 600-nm pulse with 10 nm of bandwidth (to about 50 fs).

Kafka and Baer, Opt. Lett., 12, 401 (1987)

It's best to use highly dispersive glass, like SF10, or gratings.
But compressors can still be > 1 m long.

Four-prism pulse compressor

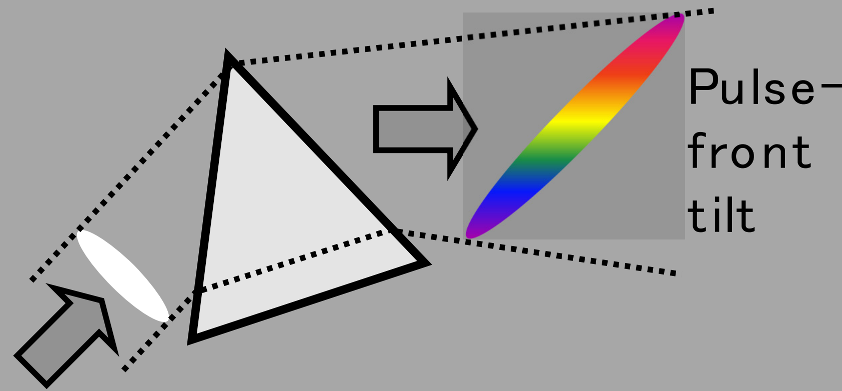
Also, alignment is critical, and many knobs must be tuned.

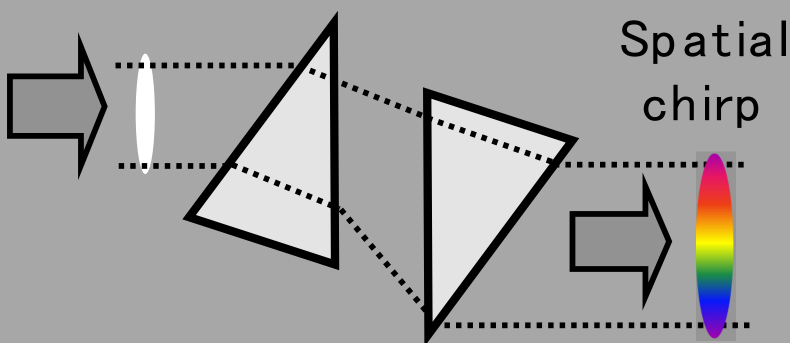


All prisms and their incidence angles must be identical.

Pulse-compressors have alignment issues.

Pulse compressors are notorious for their large size, alignment complexity, and spatio-temporal distortions.



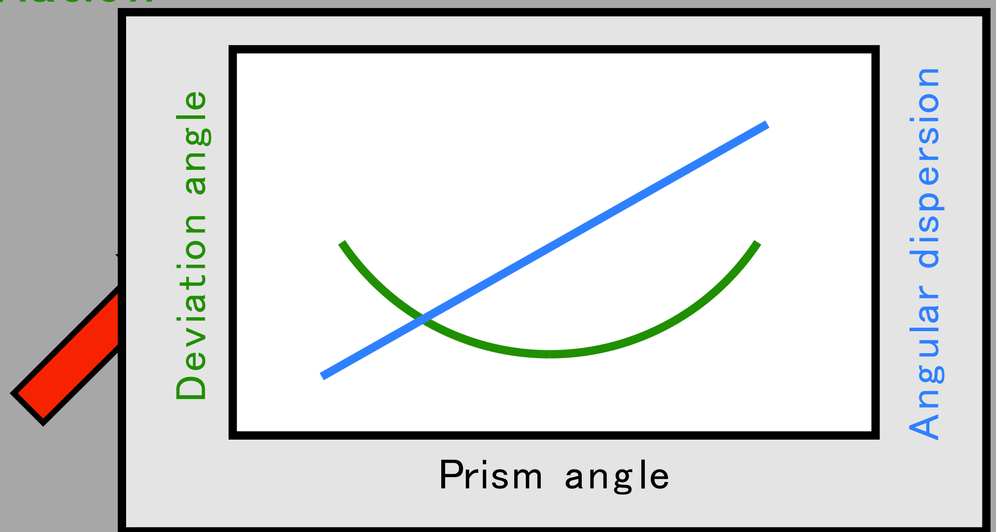


Unless the compressor is aligned perfectly, the output pulse has significant:

1. 1D beam magnification
2. Angular dispersion
3. Spatial chirp
4. Pulse-front tilt

Why is it difficult to align a pulse compressor?

The prisms are usually aligned using the **minimum deviation** condition.

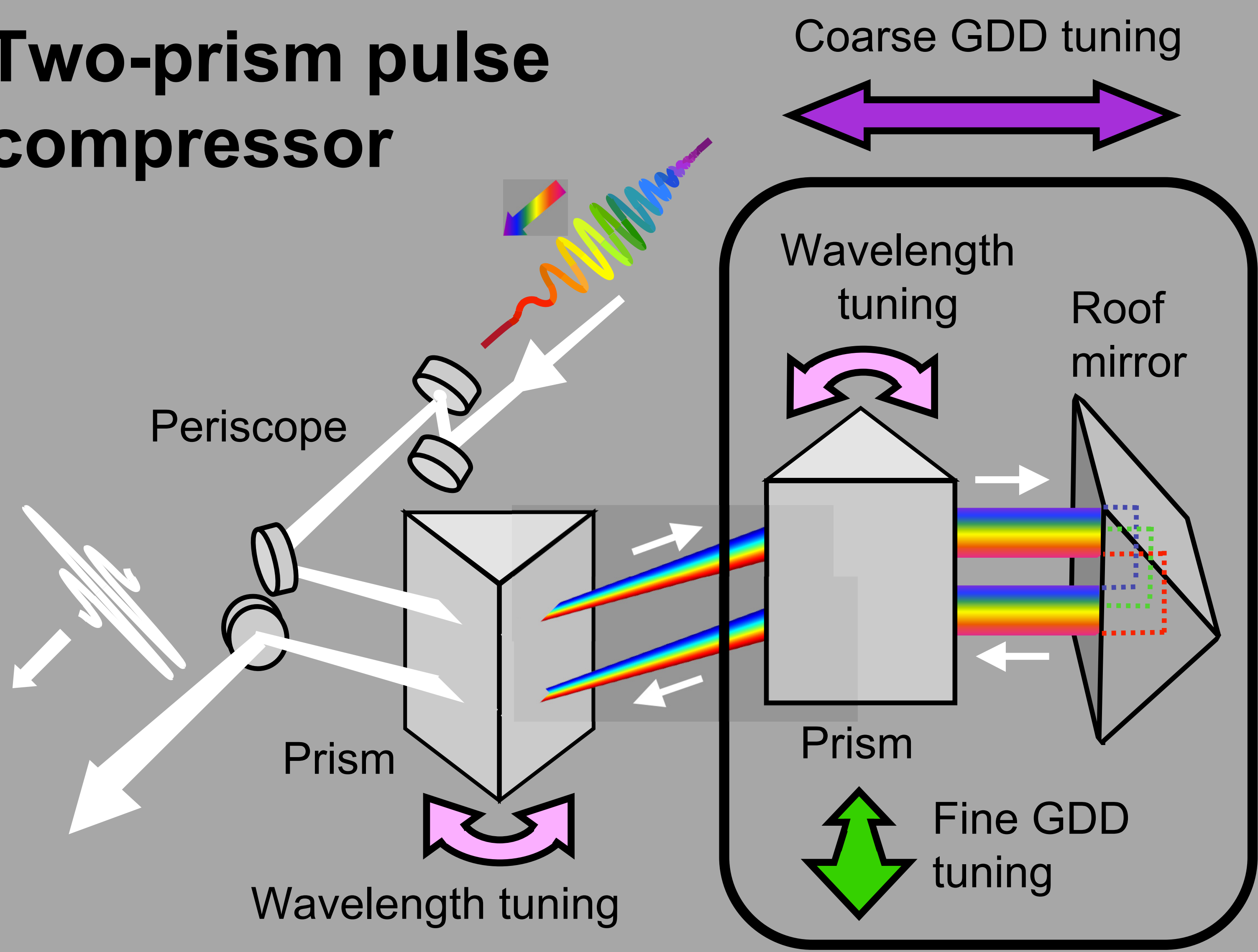


The variation of the **deviation angle** is **2nd order** in the prism angle.

But what matters is the prism **angular dispersion**, which is **1st order**!

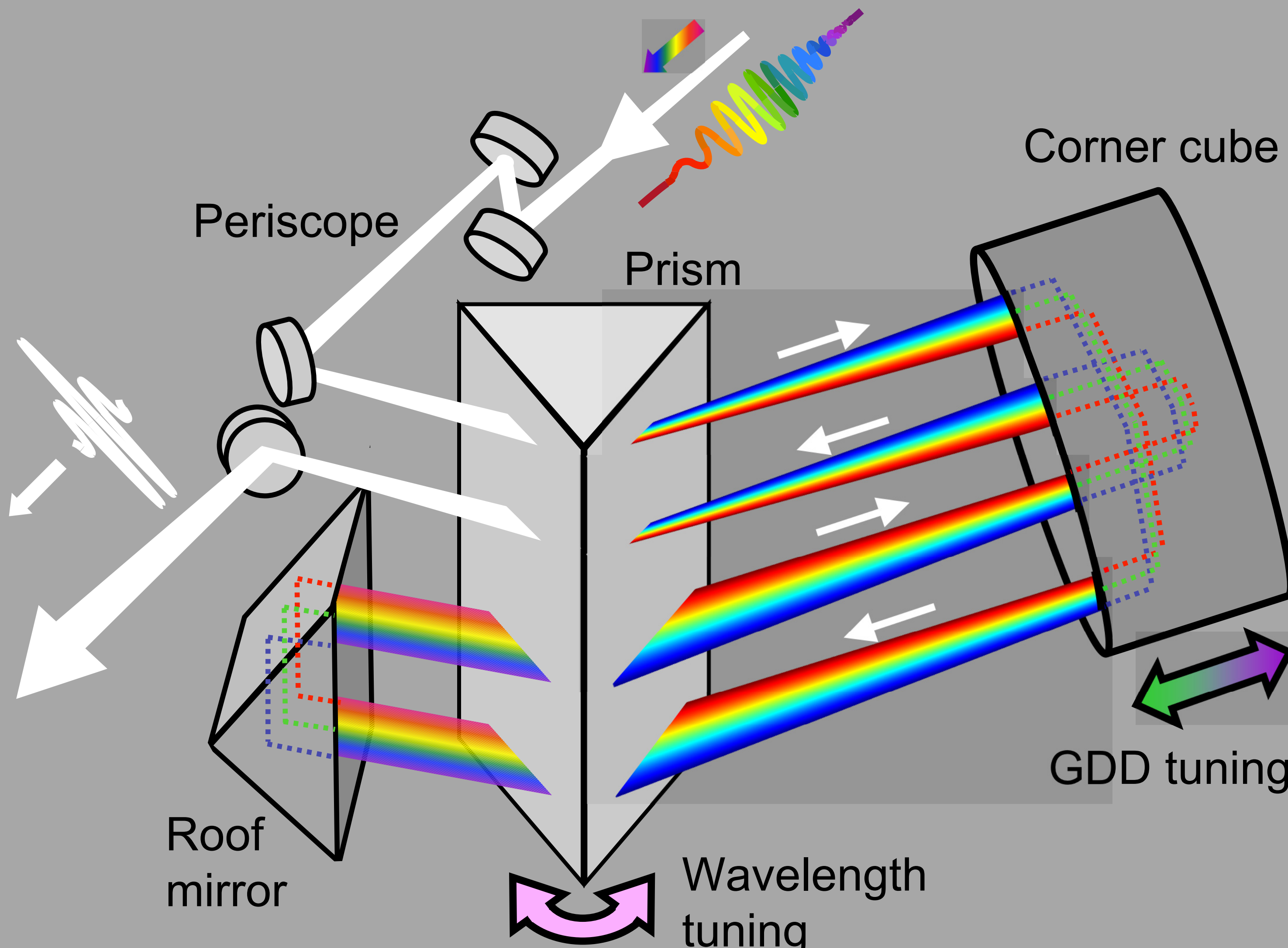
Using a 2nd-order effect to align a 1st-order effect is a bad idea.

Two-prism pulse compressor

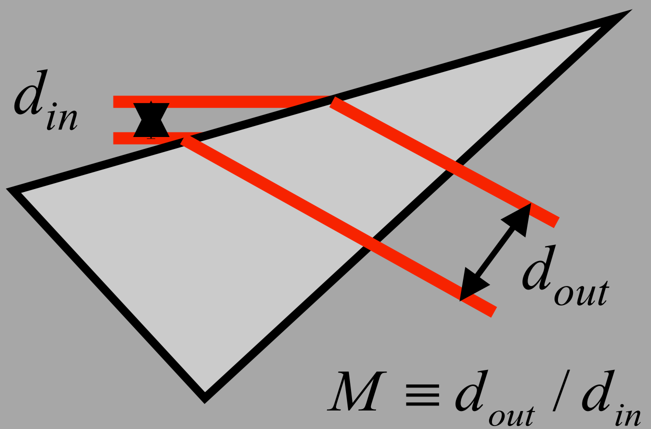


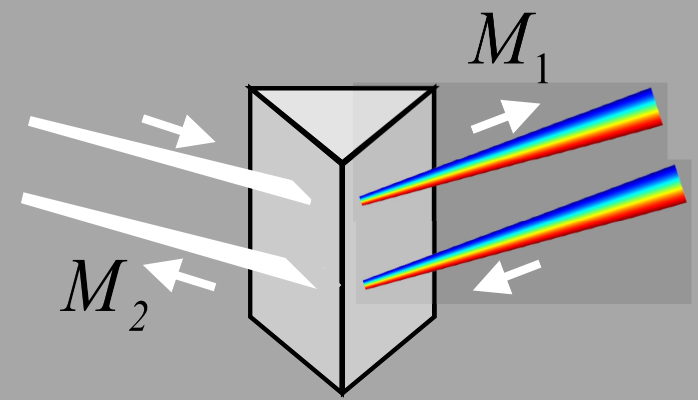
This design cuts the size and alignment issues in half.

Single-prism pulse compressor



Beam magnification is always one in a single-prism pulse compressor!



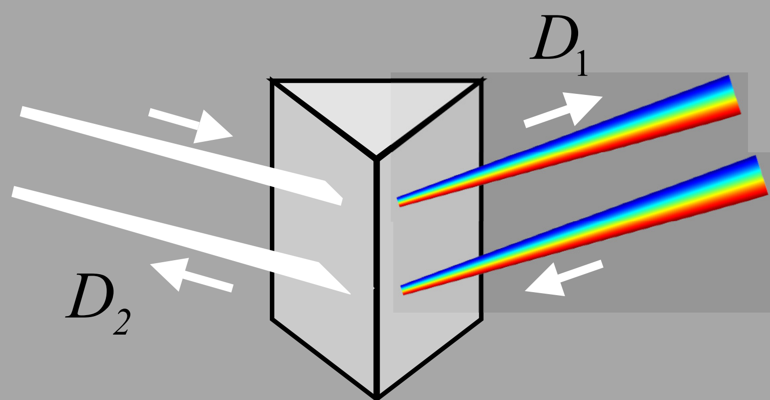


$$M_1 = \frac{1}{M_2} = M_3 = \frac{1}{M_4}$$

$$\Rightarrow \quad M_{tot} = M_1 M_2 M_3 M_4 = 1$$

The total dispersion in a single-prism pulse compressor is always zero!

The dispersion depends on the direction through the prism.



$$D_1 = -M_2 D_2 = -D_3 = M_4 D_4$$

$$D_{tot} = \frac{D_1}{M_2 M_3 M_4} + \frac{D_2}{M_3 M_4} + \frac{D_3}{M_4} + D_4 = 0$$

So the spatial chirp and pulse-front tilt are also identically zero!

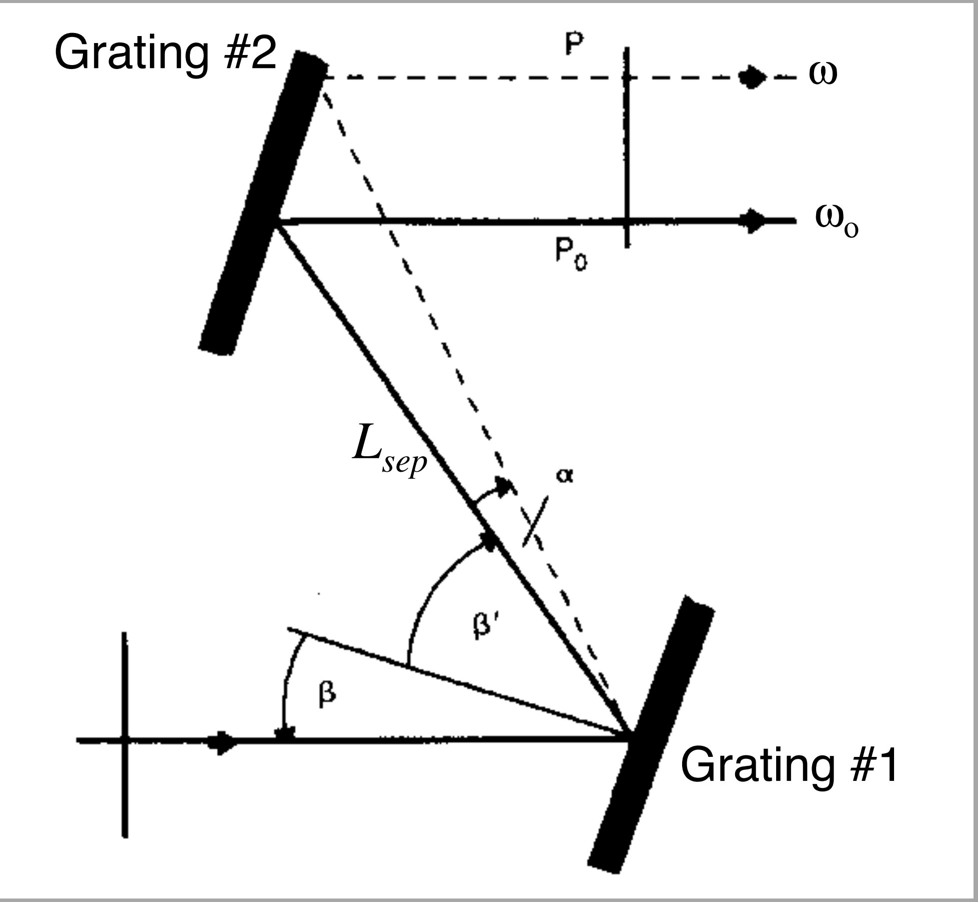
Diffraction-grating pulse compressor

The grating pulse compressor also has negative GDD.

$$\left.\frac{d^2\varphi}{d\omega^2}\right|_{\omega_0} \approx -\frac{\lambda_0^3}{2\pi c^2 d^2}\frac{L_{sep}}{\cos^2(\beta')}$$

where d = grating spacing (same for both gratings)

Note that, as in the prism pulse compressor, the larger L_{sep}, the larger the negative GDD.



2nd- and 3rd-order phase terms for prism and grating pulse compressors

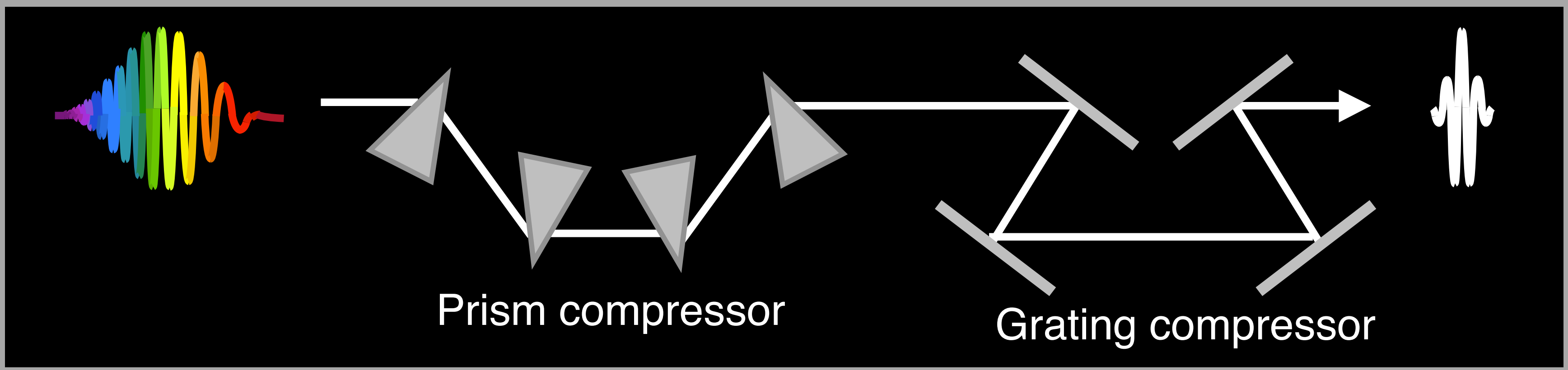
Grating compressors offer more compression than prism compressors.

Device	λ_ℓ [nm]	ω_ℓ [fs^{-1}]	φ'' [fs^{-2}]	φ''' [fs^{-3}]
SQ1 ($L = 1$ cm)	620	3.04	550	240
Piece of glass	800	2.36	362	280
Brewster prism pair, SQ1	620	3.04	-760	-1300
$\ell = 50$ cm	800	2.36	-523	-612
grating pair $b = 20$ cm; $\beta = 0°$	620	3.04	-8.2 10^4	1.1 10^5
$d = 1.2$ μm	800	2.36	-3 10^6	6.8 10^6

Note that the relative signs of the 2nd and 3rd-order terms are opposite for prism compressors and grating compressors.

Compensating 2nd and 3rd-order spectral phase

Use both a prism and a grating compressor. Since they have 3rd-order terms with opposite signs, they can be used to achieve almost arbitrary amounts of both second- and third-order phase.



Given the 2nd- and 3rd-order phases of the input pulse, φ_{input2} and φ_{input3}, solve simultaneous equations:

$$\varphi_{input2} + \varphi_{prism2} + \varphi_{grating2} = 0$$

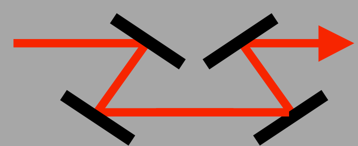
$$\varphi_{input3} + \varphi_{prism3} + \varphi_{grating3} = 0$$

This design was used by Fork and Shank at Bell Labs in the mid 1980's to achieve a 6-fs pulse, a record that stood for over a decade.

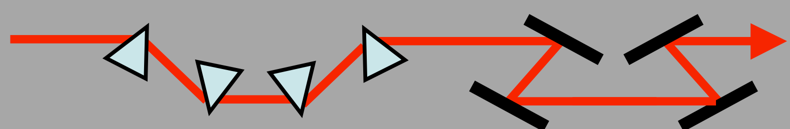
Pulse Compression Simulation

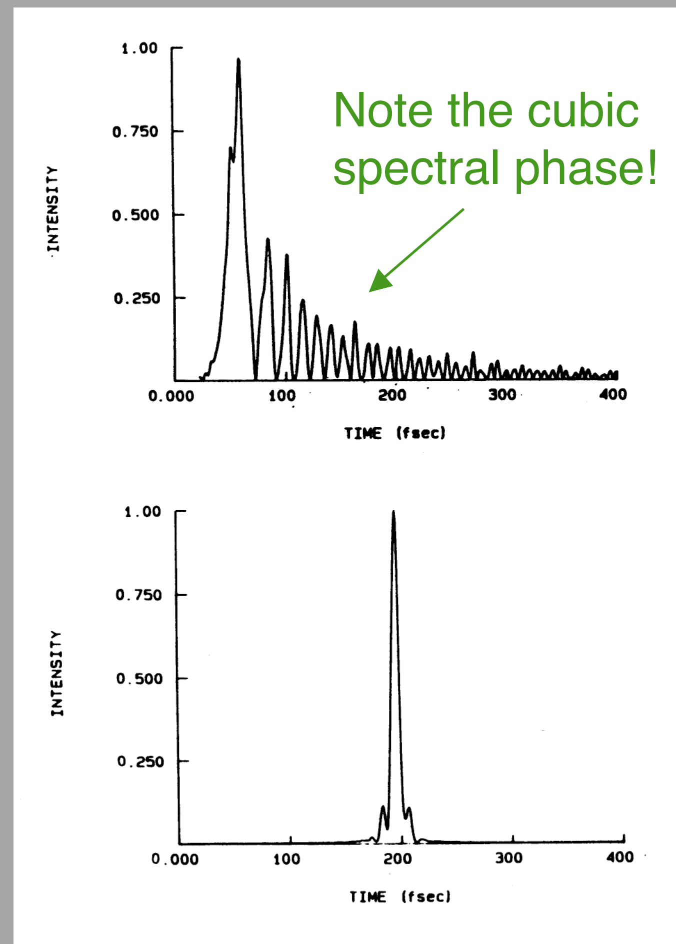
Using prism and grating pulse compressors vs. only a grating compressor

Resulting intensity vs. time
with only a grating compressor:



Resulting intensity vs. time
with a grating compressor
and a prism compressor:

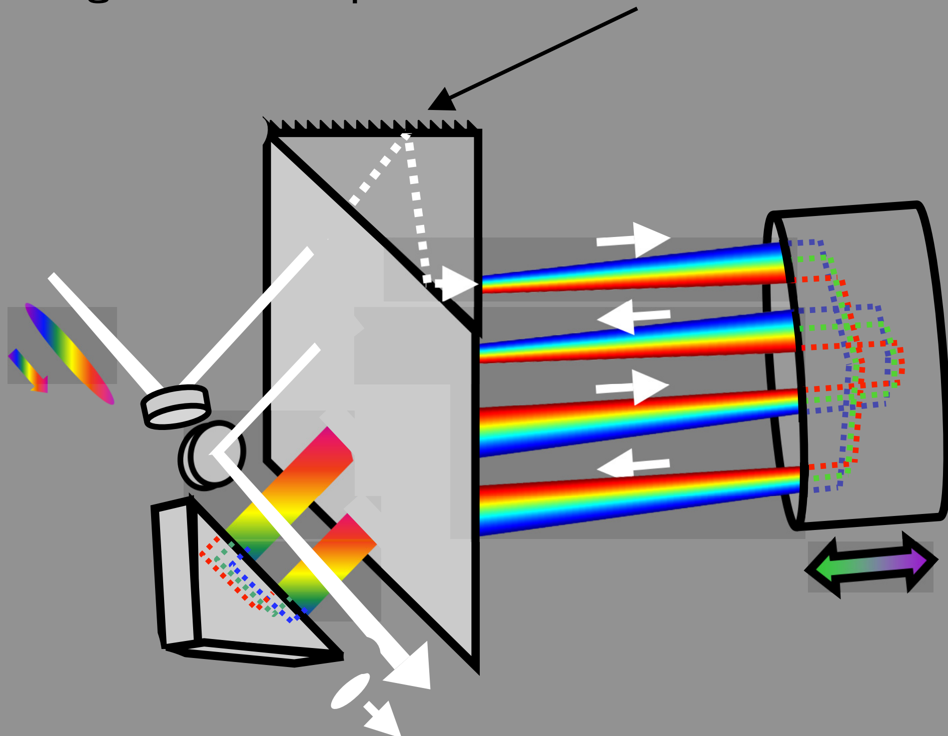




Brito Cruz, et al., Opt. Lett., 13, 123 (1988).

The grism pulse compressor has tunable third-order dispersion.

A grism is a prism with a diffraction grating etched onto it.



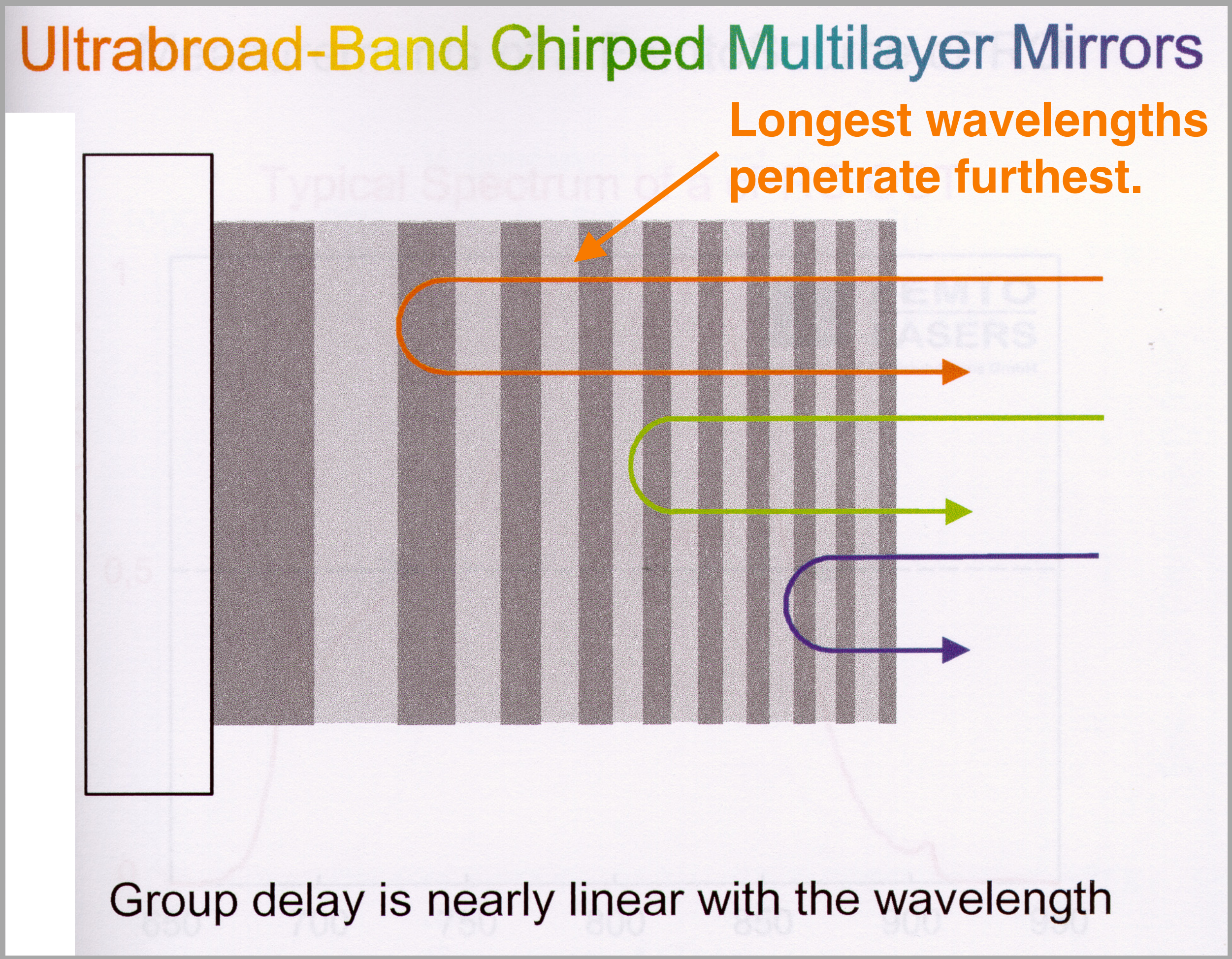
The (transmission) grism equation is:

$$a\,[\sin(\theta_m) - n\,\sin(\theta_i)] = m\lambda$$

Note the factor of n, which does not occur for a diffraction grating.

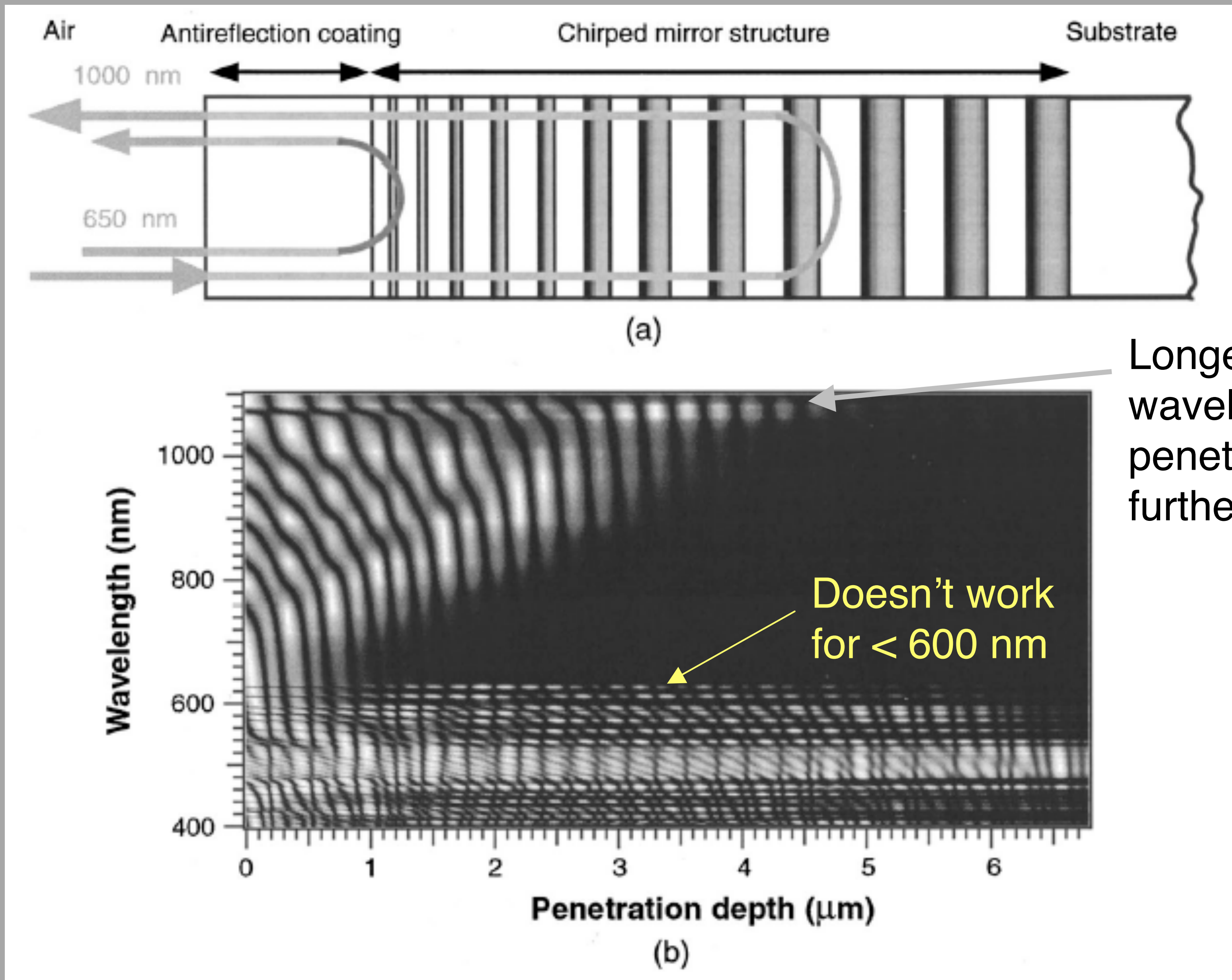
A grism compressor can compensate for both 2[nd] and 3[rd]-order dispersion due even to many meters of f ber.

Chirped mirror coatings also yield dispersion compensation.



Such mirrors avoid spatio-temporal effects, but they have limited GDD.

Chirped mirror coatings



Longest wavelengths penetrate furthest.

Doesn't work for < 600 nm