MATH332-Linear Algebra	**Homework Six**

Abstract Vector Spaces, Bases and Coordinates, Matrix Spaces

Text: Chapter 4 Section Overviews: 4.1-4.6

Quote of Homework Six	
Barron Münchausen:Your reality, sir, is lies and balderdash and I'm delighted to say that I have no grasp of it whatsoever.	
	The Adventures of Barron Münchausen : (1988)

1. Abstract Vector Spaces

1.1. **Linear Ordinary Differential Equations.** Verify that the set of all n-times continuously differentiable functions on $[a,b]$, which satisfies the homogeneous linear ordinary differential equation $L[y]=0$,

$$V=\left\{y\in C^{(n)}[a,b]:L[y]=a_n(t)\frac{d^ny}{dt^n}+a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}}+\cdots+a_0(t)y=0,\text{ where }a_0,\ldots,a_n\in C[a,b]\right\},$$

is a vector subspace of the vector space of all functions.[1]

1.2. **Polynomial Subspaces.** Prove that if H is the set of all polynomials up to degree n, such that $p(0)=0$, then H is a subspace of $\mathbb{P}_n$.

1.3. **Function Subspaces.** Prove that if $H=\{f\in C[a,b]:f(a)=f(b)\}$, then H is a subspace of $C[a,b]$.

2. Matrix Space

Given,

$$\mathbf{A}=\begin{bmatrix}-8&-2&-9\\6&4&8\\4&0&4\end{bmatrix},\quad\mathbf{w}=\begin{bmatrix}2\\1\\-2\end{bmatrix},\quad\mathbf{B}=\begin{bmatrix}2&-3&6&2&5\\-2&3&-3&-3&-4\\4&-6&9&5&9\\-2&3&3&-4&1\end{bmatrix}.\tag{1}$$

2.1. **Column Space Verification.** Is $\mathbf{w}$ in the column space of $\mathbf{A}$? That is, does $\mathbf{w}\in\text{Col }\mathbf{A}$?

2.2. **Null Space Verification.** Is $\mathbf{w}$ in the null space of $\mathbf{A}$? That is, does $\mathbf{w}\in\text{Nul }\mathbf{A}$?

2.3. **Bases for Nul B.** Determine a basis and the dimension of Nul $\mathbf{B}$.

2.4. **Bases for Col B.** Determine a basis and the dimension of Col $\mathbf{B}$.

2.5. **Bases for Row B.** Determine a basis and the dimension of Row $\mathbf{B}$.

3. Theory

Prove the following statements:

3.1. **Pivot Review.** $\dim\text{Row }\mathbf{A}+\dim\text{Nul }\mathbf{A}=n$ where $\mathbf{A}\in\mathbb{R}^{m\times n}$.

3.2. **More Pivoting.** $\text{Rank }\mathbf{A}+\dim\text{Nul }\mathbf{A}^{\mathrm{T}}=m$ where $\mathbf{A}\in\mathbb{R}^{m\times n}$.

3.3. **Dimensional Arguments.** $\mathbf{Ax}=\mathbf{b}$ has a solution for each $\mathbf{b}\in\mathbb{R}^m$ if and only if the equation $\mathbf{A}^{\mathrm{T}}\mathbf{x}=\mathbf{0}$ has only the trivial solution.[2]

3.4. **Spectral Properties of Transpositions.** The characteristic polynomial of $\mathbf{A}$ is equal to the characteristic polynomial of $\mathbf{A}^{\mathrm{T}}$. [3]

3.5. **Invertible Matrix Redux.** If $\mathbf{A}$ is an invertible matrix with eigenvalue λ then λ^{-1} is an eigenvalue of $\mathbf{A}^{-1}$.[4]

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[1]The critical idea is to show that if $u,v\in V$ then $L[c_1u+c_2v]=0$ where $c_1,c_2\in\mathbb{R}$.

[2]For the forward direction use theorem 1.4.4 on page 43 and problem 3.3 to prove that the dimension of the null space of $\mathbf{A}^{\mathrm{T}}$ is zero.

[3]Note that $\mathbf{I}$ is a symmetric matrix then use rules for the transposition of a sum and determinants of transposes.

[4]Start with $\mathbf{Ax}=\lambda\mathbf{x}$ and multiply on the left by $\mathbf{A}^{-1}$.

3.6. **Invertible Diagonalization.** If $\mathbf{A}$ is both diagonalizable and invertible, then so is $\mathbf{A}^{-1}$.[5]

3.7. **Transpositions if Diagonalization.** If $\mathbf{A}$ has n linearly independent eigenvectors, then so does $\mathbf{A}^{\mathrm{T}}$. [6]

4. Change of Bases

The standard basis for $\mathbb{R}^2$ are the column vectors, $\{\mathbf{e}_1, \mathbf{e}_2\}$ of $\mathbf{I}_{2\times 2}$. In class we looked at the basis $\mathfrak{B} = \{[1,1]^{\mathrm{T}}, [-1,1]^{\mathrm{T}}\}$. This basis is rotated $\frac{\pi}{4}$ radians counter-clockwise from the standard basis and does not preserve the notion of length from the standard coordinate system.

4.1. **Rotations Revisited.** Determine a basis for $\mathbb{R}^2$, which is rotated $\frac{\pi}{4}$ radians counter-clockwise from the standard basis and preserves the unit length associated with the standard basis.

4.2. **Orthogonal Coordinates.** Show that, for this basis, the change-of-coordinates matrix $\mathbf{P}_{\mathfrak{B}}$ is such that, $\mathbf{P}_{\mathfrak{B}}\mathbf{P}_{\mathfrak{B}}^{\mathrm{T}} = \mathbf{P}_{\mathfrak{B}}^{\mathrm{T}}\mathbf{P}_{\mathfrak{B}} = \mathbf{I}_{2\times 2}$.

4.3. **Coordinate Changes.** Given that $[\mathbf{x}_1]_{\mathfrak{B}} = [\sqrt{2}, \sqrt{2}]^{\mathrm{T}}$ determine $\mathbf{x}_1$ and given that $\mathbf{x}_2 = \left[\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right]^{\mathrm{T}}$ determine $[\mathbf{x}_2]_{\mathfrak{B}}$. Calculate the magnitude of both of the vectors previously calculated.

4.4. **Polynomial Spaces.** The Hermite polynomials are a sequence of orthogonal polynomials, which arise in probability, combinatorics and physics.[7] The first four polynomials in this sequence are given as,

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = -2 + 4x^2, \quad H_3(x) = -12x + 8x^3, \quad x \in (-\infty, \infty).$$

4.5. **Linear Independence.** Show that $\mathfrak{B} = \{1, 2x, -2 + 4x, -12x + 8x^3\}$ is a basis for $\mathbb{P}_3$.
Hint: Determine the coordinate vectors of the Hermite polynomials relative to the standard basis.

4.6. **Change of Basis.** Let $\mathbf{p}(x) = 7 - 12x - 8x^2 + 12x^3$. Find the coordinate vector of $\mathbf{p}$ relative to $\mathfrak{B}$.
Hint: Determine $\{c_0, c_1, c_2, c_3\}$ such that $\mathbf{p}(x) = \sum_{i=0}^{3} c_i H_i(\mathrm{x})$.

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[5] Note that if $\mathbf{D}$ is a diagonal matrix then $\mathbf{D}^{-1}$ is the matrix whose diagonal elements are scalar inverses of the diagonal elements of $\mathbf{D}$.

[6] Use theorem 5.3.5 and the fact that if $\mathbf{P}$ is invertible then $(\mathbf{P}^{\mathrm{T}})^{-1} = (\mathbf{P}^{-1})^{\mathrm{T}}$. It is also useful to note that diagonal matrices are symmetric.

[7] In physics these polynomials manifest as the spatial solutions to Schrödinger's wave equation under a harmonic potential, which evolves the probability distribution of a quantum mechanical particle near an energy minimum. As it turns out there are infinitely-many Hermite polynomials and consequently one can show that this particle has infinitely-many allowed quantized energy levels, which are evenly spaced. In probability they arise as different moments of a standard normal distribution.