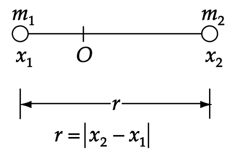
8-6.



When two particles are initially at rest separated by a distance r_0, the system has the total energy

$$E_0 = -G\,\frac{m_1 m_2}{r_0} \tag{1}$$

The coordinates of the particles, x_1 and x_2, are measured from the position of the center of mass. At any time the total energy is

$$E = \frac{1}{2}\,m_1\dot{x}_1^2 + \frac{1}{2}\,m_2\dot{x}_2^2 - G\,\frac{m_1 m_2}{r} \tag{2}$$

and the linear momentum, at any time, is

$$p = m_1\dot{x}_1 + m_2\dot{x}_2 = 0 \tag{3}$$

From the conservation of energy we have $E = E_0$, or

$$-G\,\frac{m_1 m_2}{r_0} = \frac{1}{2}\,m_1\dot{x}_1^2 + \frac{1}{2}\,m_2\dot{x}_2^2 - G\,\frac{m_1 m_2}{r} \tag{4}$$

Using (3) in (4), we find

$$\boxed{\begin{aligned} \dot{x}_1 &= v_1 = m_2\sqrt{\frac{2G}{M}\left[\frac{1}{r} - \frac{1}{r_0}\right]} \\ \dot{x}_1 &= v_2 = -m_1\sqrt{\frac{2G}{M}\left[\frac{1}{r} - \frac{1}{r_0}\right]} \end{aligned}} \tag{5}$$

8-10. For circular motion

$$T = \frac{1}{2}\,m_e\omega^2\,r_e^2$$

$$U = -\frac{GM_s\,m_e}{r_e}$$

We can get ω^2 by equating the gravitational force to the centripetal force

$$\frac{GM_s\,m_e}{r_e^2} = m_e\omega^2\,r_e$$

or

$$\omega^2 = \frac{GM_s}{r_e^3}$$

So

$$T = \frac{1}{2} m_e r_e^2 \cdot \frac{GM_s}{r_e^3} = \frac{GM_s\, m_e}{2r_e} = -\frac{1}{2}U$$

$$E = T + U = \frac{1}{2}U$$

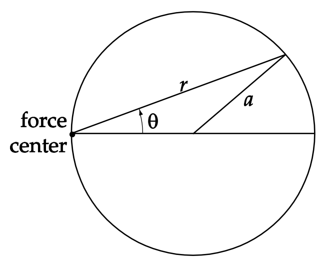
If the sun's mass suddenly goes to $\frac{1}{2}$ its original value, T remains unchanged but U is halved.

$$E' = T' + U' = T + \frac{1}{2}U = -\frac{1}{2}U + \frac{1}{2}U = 0$$

The energy is 0, so the orbit is a parabola. For a parabolic orbic, the earth will escape the solar system.

8-11. For central-force motion the equation of orbit is [Eq. (8.21)]

$$\frac{d^2}{d\theta^2}\left[\frac{1}{r}\right] + \frac{1}{r} = -\frac{\mu r^2}{\ell^2} F(r) \tag{1}$$



In our case the equation of orbit is

$$r = 2a\cos\theta \tag{2}$$

Therefore, (1) becomes

$$\frac{1}{2a}\frac{d^2}{d\theta^2}\left[(\cos\theta)^{-1}\right] + \frac{1}{2a}(\cos\theta)^{-1} = -\frac{4a^2\mu}{\ell^2} F(r)\cos^2\theta \tag{3}$$

But we have

$$\frac{d^2}{d\theta^2}\left[(\cos\theta)^{-1}\right] = \frac{d}{d\theta}\left[\frac{\sin\theta}{\cos^2\theta}\right]$$

$$= \frac{1}{\cos\theta} + \frac{2\sin^2\theta}{\cos^3\theta} \tag{4}$$

Therefore, we have

$$\frac{1}{\cos\theta} + \frac{2\sin^2\theta}{\cos^3\theta} + \frac{1}{\cos\theta} = -\frac{8a^3\,\mu}{\ell^2}F(r)\cos^2\theta \tag{5}$$

or,

$$F(r) = -\frac{\ell^2}{8a^3\mu}\frac{2}{\cos^5\theta} = -\frac{8a^2\ell^2}{\mu}\frac{1}{r^5} \tag{6}$$

so that

$$\boxed{F(r) = -\frac{k}{r^5}} \tag{7}$$

8-13. Setting $u \equiv 1/r$ we can write the force as

$$F = -\frac{k}{r^2} - \frac{\lambda}{r^3} = -ku^2 - \lambda u^3 \tag{1}$$

Then, the equation of orbit becomes [cf. Eq. (8.20)]

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{\ell^2}\frac{1}{u^2}\left(-ku^2 - \lambda u^3\right) \tag{2}$$

from which

$$\frac{d^2u}{d\theta^2} + u\left[1 - \frac{\mu\lambda}{\ell^2}\right] = \frac{\mu k}{\ell^2} \tag{3}$$

or,

$$\frac{d^2u}{d\theta^2} + \left[1 - \frac{\mu\lambda}{\ell^2}\right]\left[u - \frac{\mu k}{\ell^2}\frac{1}{1 - \frac{\mu\lambda}{\ell^2}}\right] = 0 \tag{4}$$

If we make the change of variable,

$$v = u - \frac{\mu k}{\ell^2}\frac{1}{1 - \frac{\mu\lambda}{\ell^2}} \tag{5}$$

we have

$$\frac{d^2v}{d\theta^2} + \left[1 - \frac{\mu\lambda}{\ell^2}\right]v = 0 \tag{6}$$

or,

$$\frac{d^2v}{d\theta^2} + \beta^2 v = 0 \tag{7}$$

where $\beta^2 = 1 - \mu\lambda/\ell^2$. This equation gives different solutions according to the value of λ. Let us consider the following three cases:

i) $\lambda < \ell^2/\mu$:

For this case $\beta^2 > 0$ and the solution of (7) is

$$v = A\cos(\beta\theta - \delta)$$

By proper choice of the position $\theta = 0$, the integration constant δ can be made to equal zero. Therefore, we can write

$$\boxed{\frac{1}{r} = A\cos\beta\theta + \frac{\mu k}{\ell^2 - \mu\lambda}} \tag{9}$$

When $\beta = 1$ ($\lambda = 0$), this equation describes a conic section. Since we do not know the value of the constant A, we need to use what we have learned from Kepler's problem to describe the motion. We know that for $\lambda = 0$,

$$\frac{1}{r} = \frac{\mu k}{\ell^2}(1 + \varepsilon\cos\theta)$$

and that we have an ellipse or circle ($0 \le \varepsilon < 1$) when $E < 1$, a parabola ($\varepsilon = 1$) when $E = 0$, and a hyperbola otherwise. It is clear that for this problem, if $E \ge 0$, we will have some sort of parabolic or hyperbolic orbit. An ellipse should result when $E < 0$, this being the only bound orbit. When $\beta \ne 1$, the orbit, whatever it is, precesses. This is most easily seen in the case of the ellipse, where the two turning points do not have an angular separation of π. One may obtain most constants of integration (in particular A) by using Equation (8.17) as a starting point, a more formal approach that confirms the statements made here.

ii) $\lambda = \ell^2/\mu$

For this case $\beta^2 = 0$ and (3) becomes

$$\frac{d^2u}{d\theta^2} = \frac{\mu k}{\ell^2} \tag{10}$$

so that

$$\boxed{u = \frac{1}{r} = \frac{\mu k}{2\ell^2}\theta^2 + A\theta + B} \tag{11}$$

from which we see that r continuously decreases as θ increases; that is, the particle spirals in toward the force center.

iii) $\lambda > \ell^2/\mu$

For this case $\beta^2 < 0$ and the solution (7) is

$$v = A\cosh\left(\sqrt{-\beta^2}\ \theta - \delta\right) \tag{12}$$

δ may be set equal to zero by the proper choice of the position at which $\theta = 0$. Then,

$$\boxed{\frac{1}{r} = A\cosh\left(\sqrt{-\beta^2}\ \theta\right) + \frac{\mu k}{\ell^2 - \mu\lambda}} \tag{13}$$

Again, the particle spirals in toward the force center.