Note Title 1/18/2008

Review on ∞-dimensional vector spaces. See also, class from 10/22/07 in 311 (on the 311 wiki)

$$\vec{z} = a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3 + \cdots a_N \hat{e}_N$$

$$\vec{z} \in \mathbb{R}^N$$

$$\|\vec{z}\|^2 = \vec{z} \cdot \vec{z} =$$

$$(a_1 \hat{e}_1 + a_2 \hat{e}_2 \cdots) \cdot (a_1 \hat{e}_1 + a_2 \hat{e}_2 \cdots)$$

$$= |a_1|^2 \hat{e}_1 \cdot \hat{e}_1 + |a_2|^2 \hat{e}_2 \cdot \hat{e}_2 \cdots$$ [diagonal terms]

$$+ \; a_1 a_2 \hat{e}_1 \cdot \hat{e}_2 \cdots$$ [cross terms]

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij} \quad \left\{ \begin{array}{ll} \text{i.e } 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{array} \right\}$$

Kronecker Delta

$$\Rightarrow \|\vec{z}\|^2 = \sum_{i=1}^{N} |a_i|^2$$

Now consider if $N \to \infty$

$$\vec{z} = \sum_{i=1}^{\infty} a_i \hat{e}_i$$

where still $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$

$$\boxed{\|z\|^2 = \sum_{i=1}^{\infty} |a_i|^2}$$

not obvious if this is finite: convergence is an issue for infinite vectors.

The set of all vectors $z \in \mathbb{R}^{\infty}$ such that $\|z\|^2 < \infty$ is called

l_2. "little l-two"

Now consider a function and its Fourier series:

$$f(x) = \sum_{l=-\infty}^{\infty} c_n e^{in\pi x/l}$$

Here f is assumed to be periodic on $[-l, l]$.

in 311 we defined the inner prod of 2 functions on $[-l, l]$ as

$$(f,g) = \int_{-\ell}^{\ell} f(x)\, g^*(x)\, dx$$

so $\quad \|f\|^2 = (f, f^*)$

$$= \int_{-\ell}^{\ell} |f(x)|^2 dx$$

$$= \int_{-\ell}^{\ell} \left[\sum_{j=-\infty}^{\infty} c_j e^{i\pi j x/\ell}\right]\left[\sum_{k=-\infty}^{\infty} c_k^* e^{-i\pi k x/\ell}\right] dx$$

$$= \sum_j \sum_k c_j c_k^* \underbrace{\int_{-\ell}^{\ell} e^{i\pi(j-k)x/\ell}\, dx}_{\delta_{jk}}$$

$$= \sum_{j=-\infty}^{\infty} |c_j|^2$$

Same issue as before. in order that $\|f\|$ be finite we require that

$$\sum_{j=-\infty}^{\infty} |c_j|^2 < \infty$$

be a convergent sequence.

In fact a function that can be represended by a fourier series is equivalent to a vector in l_p, since it is completely specified by its sequence of coefficients.

$\boxed{\text{So functions } \underline{\text{are}} \text{ vectors}}$

$$\vec{x}, \vec{y} \in \mathbb{R}^N \qquad \vec{x}+\vec{y} = \sum_{i=1}^{N} (x_i + y_i)\hat{e}_i$$

"component-wise addition

$f(x) + g(x) = h(x)$. At each point $\tilde{x}$ in the domain, we say

$$h(\tilde{x}) = f(\tilde{x}) + g(\tilde{x}) \quad \text{like for vectors}$$

similarly $\quad \alpha(\vec{x}+\vec{y}) = \alpha\vec{x} + \alpha\vec{y}$

$$\alpha(f(x)+g(x)) = \alpha f(x) + \alpha g(x)$$

Operators

$\boxed{\text{An operator maps vectors into vectors}}$

$A \in \mathbb{R}^{3\times 2}$

A maps vectors in $\mathbb{R}^2$ into $\mathbb{R}^3$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

E.g. operator on functions of 1 variable.

$$\frac{d}{dx} : f(x) \to f'(x)$$

$$x^2 \to 2x$$
$$1 \to 0$$
$$\frac{1}{x} \to -1/x^2$$

maps functions into functions

Last time, we showed that

PDF $= |\psi|^2 = \psi^* \psi$

So $\langle x \rangle = \int x \, \psi^* \psi \, dx$

$$\boxed{\langle x \rangle = \int \psi^* \, x \, \psi \, dx}$$

Then by differentiating $\frac{d}{dt}$ and using Schrödinger

$$\frac{d}{dt} \langle x \rangle = -\frac{i\hbar}{m} \int \psi^* \frac{\partial \psi}{\partial x} \, dx$$

Hence $m \frac{d\langle x \rangle}{dt} = -i\hbar \int \psi^* \frac{\partial \psi}{\partial x} \, dx$

$$= \int \psi^* \left(-i\hbar \frac{\partial}{\partial x}\right) \psi \, dx$$

if we in terpret

$m\frac{d\langle x\rangle}{dt}$ as the expectation

of the QM momentum, then

$$\langle p\rangle = \int \psi^* \left(-i\hbar\frac{\partial}{\partial x}\right)\psi \, dx$$

major step we say that

the operator $-i\hbar\frac{\partial}{\partial x}$

represents p in the coordinate representation

Later we will see that

$p \leftrightarrow x$ are fourier transform pairs we can now take classical formulae involving x + p and compute quantum mech expectations.

Eg. Kinetic Energy, $T = \frac{p^2}{2m}$

$$\langle T \rangle = \int \psi^* \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right) \psi \, dx$$

$$= \frac{-\hbar^2}{2m} \int \psi^* \frac{\partial^2 \psi}{\partial x^2} \, dx$$

Eg. deBroglie wave

$$\psi = \psi_0 e^{i(px - Et)/\hbar}$$

$$\psi^* = \psi_0^* e^{-i(px - Et)/\hbar}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{-p^2}{\hbar^2} \psi$$

$$\langle T \rangle = \frac{p^2}{2m} \int \psi^* \psi \, dx = \frac{p^2}{2m}$$

Because the deBroglie wave was constructed to have a precise p.

In General take any function of x, p $\quad Q(x,p)$ then

$$\langle Q(x,p)\rangle = \int \psi^* \, Q(x, -i\hbar \tfrac{\partial}{\partial x}) \, \psi \, dx$$

Ex. $\quad \psi(x,t) = \frac{1}{(2\pi)^{1/4}} e^{-x^2/4}$ $\quad \left\{ \begin{array}{l} \langle x\rangle = 0 \\ \sigma = 1 \end{array} \right.$

Remember: it is $|\psi|^2$ that is <u>normalized</u> not ψ!

$$|\psi|^2 = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad \checkmark$$

$$\frac{\partial\psi}{\partial x} = \frac{1}{(2\pi)^{1/4}} \frac{x}{2} e^{-x^2/4}$$

$$\langle p\rangle = \frac{-i\hbar}{2\sqrt{2\pi}} \int x e^{-x^2/2} dx = 0$$

$$\langle p^2\rangle = \frac{-\hbar^2}{2\sqrt{2\pi}} \int \left[\frac{x^2}{2} - 1\right] e^{-x^2/2} dx$$

$$= \frac{\hbar^2}{2\sqrt{2\pi}} \sqrt{\frac{\pi}{2}} = \frac{\hbar^2}{4}$$