HW5 Solutions

Note Title 10/7/2007

3-11

④ $C = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ The inverse of this is a rotation by $-\theta$

$\cos(-\theta) = \cos\theta \qquad \sin(-\theta) = -\sin\theta$

so $C^{-1} =$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

or

$$\left(\begin{array}{cc|cc} \cos\theta & \sin\theta & 1 & 0 \\ -\sin\theta & \cos\theta & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} \sin\theta\cos\theta & \sin^2\theta & \sin\theta & 0 \\ -\sin\theta\cos\theta & \cos^2\theta & 0 & \cos\theta \end{array}\right)$$

$$\left(\begin{array}{cc|cc} \sin\theta\cos\theta & \sin^2\theta & \sin\theta & 0 \\ 0 & 1 & \sin\theta & \cos\theta \end{array}\right)$$

$$\left(\begin{array}{cc|cc} \cos\theta & \sin\theta & 1 & 0 \\ 0 & 1 & \sin\theta & \cos\theta \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & \tan\theta & 1/\cos\theta & 0 \\ 0 & 1 & \sin\theta & \cos\theta \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & 0 & \cos\theta & -\sin\theta \\ 0 & 1 & \sin\theta & \cos\theta \end{array}\right) \checkmark$$

6.

$$C = \begin{array}{cc} \vec{r}_1 & \vec{r}_2 \\ \multicolumn{2}{c}{\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}} \end{array} = \begin{pmatrix} \vec{r}_1 & \vec{r}_2 \end{pmatrix}$$

given that $\|\vec{r}_1\| = \|\vec{r}_2\| = 1$ $\quad (\vec{r}_1, \vec{r}_2) = 0$.

$$C^T = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = \begin{pmatrix} \vec{r}_1^{\,T} \\ \vec{r}_2^{\,T} \end{pmatrix}$$

$$C^T C = \begin{pmatrix} \vec{r}_1 \cdot \vec{r}_1 & \vec{r}_1 \cdot \vec{r}_2 \\ \vec{r}_1 \cdot \vec{r}_2 & \vec{r}_2 \cdot \vec{r}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= CC^T$$

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a) $\quad \text{Det}\,(C^{-1} M C)$

$$\text{Det}\,(C^{-1} M C) = \text{Det}(C^{-1})\,\text{Det}(M)\,\text{Det}(C)$$

since $\quad \text{Det}\,(C^{-1} C) = \text{Det}\,(I) = 1$

$$\text{Det}\,(C^{-1})\,\text{Det}\,(C) = 1$$

$$\Rightarrow \text{Det}\,(C^{-1}) = \frac{1}{\text{Det}(C)}$$

$$\Rightarrow \text{Det}\,(C^{-1} M C) = \text{Det}\,(M)$$

if C is the matrix of Σ-vectors of M, then

$$C^{-1} M C = D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \lambda_N \end{bmatrix}$$

$$\mathrm{Det}(D) = \lambda_1 \times \lambda_2 \times \cdots \lambda_N$$

$$\Rightarrow \mathrm{Det}(M) = \lambda_1 \times \lambda_2 \cdots \lambda_N$$

10 $\quad \mathrm{Tr}(C^{-1} M C)$

9:13 says $\mathrm{Tr}(C^{-1} M C) =$

$$\mathrm{Tr}(M \underbrace{C C^{-1}}_{I})$$

$$= T(M)$$

But since $C^{-1} M C = D$

$$\mathrm{Tr}(M) = \mathrm{Tr}(D) = \lambda_1 + \lambda_2 + \cdot \lambda_N$$

(33)

$$M = \begin{pmatrix} A & H \\ H & B \end{pmatrix}$$

char. polynomial $= (A-\lambda)(B-\lambda) - H^2 = 0$

$= \lambda^2 - \lambda(A+B) + AB - H^2 = 0$

$$\lambda_\pm = \frac{A+B}{2} \pm \frac{1}{2}\underbrace{\sqrt{(A+B)^2 - 4AB + 4H^2}}_{\sqrt{(A-B)^2 + 4H^2}}$$

since the argument of the $\sqrt{\ }$ is nonnegative, the E-values are real.

$$\begin{pmatrix} A & H \\ H & B \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda_\pm \begin{pmatrix} x \\ y \end{pmatrix}$$

$Ax + Hy = \lambda x$ $\quad (A-\lambda)x = -Hy$

$Hx + By = \lambda y$ $\quad (B-\lambda)y = -Hx$

put $\lambda = \lambda_1$ in the first to get

$x = \dfrac{-H}{A-\lambda_1} y$, similarly $y = \dfrac{-H}{B-\lambda_2} x$

$$\vec{r}_1 \qquad\qquad \vec{r}_2$$

$$\begin{pmatrix} -\frac{H}{A-\lambda_1} \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -\frac{H}{B-\lambda_2} \end{pmatrix}$$ are the 2

Eigenvectors.

$$\vec{r}_1 \cdot \vec{r}_2 = -\frac{H}{A-\lambda_1} - \frac{H}{B-\lambda_2}$$

$$= -H\left[\frac{(B-\lambda_2) + (A-\lambda_1)}{(A-\lambda_1)(B-\lambda_2)}\right]$$

$$= -H\left[\frac{\overbrace{(B+A)}^{\text{trace}} - \overbrace{(\lambda_1+\lambda_2)}^{\text{Trace}}}{(A-\lambda_1)(B-\lambda_2)}\right] \equiv 0$$

<u>NB</u> 11.24 is more general.

if you simply appeal to 11.24

I will accept that.

(37) a) $H\vec{r}_1 = \lambda_1 \vec{r}_1$ b) $H\vec{r}_2 = \lambda_2 \vec{r}_2$

$H^+ = H$ by assumption

a) $\Rightarrow$ $\vec{r}_1^{\,+} H^+ = \lambda_1^* \vec{r}_1^{\,+} = \lambda_1 \vec{r}_1^{\,+}$ by prev. result.
$= \vec{r}_1^{\,+} H$

$$\Rightarrow \vec{r}_1^{\,+} \underbrace{H \vec{r}_2}_{\lambda_2 \vec{r}_2} = \lambda_1 \vec{r}_1^{\,+} \vec{r}_2$$

$$\Rightarrow \lambda_2 \vec{r}_1^{\,+} \vec{r}_2 = \lambda_1 \vec{r}_1^{\,+} \vec{r}_2$$

$$\Rightarrow (\lambda_2 - \lambda_1) \vec{r}_1^{\,+} \vec{r}_2$$

so either $\lambda_1 = \lambda_2$ or $\vec{r}_1$ is orthogonal to $\vec{r}_2$

(57) $$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \ddots \; \lambda_N \end{bmatrix}$$

$$D^2 = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & & & \\ & \lambda_2^2 & & \\ & & \ddots & \\ & & & \lambda_N^2 \end{bmatrix}$$

in general $D^m = \begin{bmatrix} \lambda_1^m & & & \\ & \lambda_2^m & & \\ & & \ddots & \\ & & & \lambda_N^m \end{bmatrix}$.

$$D = C^{-1} M C$$

$$D^2 = (C^{-1} M \underbrace{C)(C^{-1}}_{I} M C)$$

$$= C^{-1} M^2 C$$

$$D^3 = D^2 D = C^{-1} M^2 \underbrace{C C^{-1}}_{I} M C$$

$$= C^{-1} M^3 C$$

$$\vdots$$

$$\boxed{D^N = C^{-1} M^N C}$$