Advanced Engineering Mathematics		**Homework Six**
Differential Equations : Sturm-Liouville Problems, Power Series, Conservations Laws, PDE		
Text: 5.1, 5.7, 12.1	Lecture Notes : ODE Review and 13	Lecture Slides: N/A

Quote of Homework Seven Solutions
You pick the place and I'll choose the time and I'll climb that hill in my own way. Just wait a while, for the right day and as I rise above the treeline and the clouds I look down and hear the sound of the things you said today.
Pink Floyd : Fearless - Meddle (1971)

1. Sturm-Liouville Problems

Recall the Sturm-Liouville eigenproblem given by,

$$Lu = \frac{1}{w(x)}\left(-\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u\right) = \lambda u, \ \lambda \in \mathbb{C} \tag{1}$$

whose nontrivial eigenfunctions must satisfy the boundary conditions,

$$k_1 u(a) + k_2 u'(a) = 0 \tag{2}$$

$$l_1 u(b) + l_2 u'(b) = 0. \tag{3}$$

1.1. **Orthogonality of Solutions.** Let (λ_1, u_1) and (λ_2, u_2) be two different eigenvalue/eigenfunction pairs. Show that u_1 and u_2 are orthogonal. That is, show that $\langle u_1, u_2\rangle = 0$ with respect to the inner-product defined by $\langle f, g\rangle = \int_a^b f(x)g(x)w(x)dx$.

Before we begin with this derivation we recall the following from linear algebra.

- If a square matrix is self-adjoint then it is possible to use its eigenvectors to form an orthogonal basis for $\mathbb{R}^n$ and using this basis it is possible to construct the orthogonal diagonalization $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{U}^{\mathrm{H}}$, where $\mathbf{D} \in \mathbb{R}^{n\times n}$.

This can be distilled into,

- If $\mathbf{A}$ is self-adjoint then its eigenvalues are real and eigenvectors manifesting from different eigenvalues are orthogonal.

This is a terribly important result in the theory of linear algebra and generally the theory of linear transformations. We explore this result in the context of linear differential operators given by (1)-(3). Though things will get a little intense I refer the reader back to the previous bullet point for some modicum of comfort.

We need to show that eignfunctions arising from different eigenvalues are orthogonal under the abstract inner-product given in the problem. First we now that for the previous eigenpairs we have from (1),[1]

$$(p(x)u_1)' + [\lambda_1 w(x) - q(x)]\, u_1 = 0, \tag{4}$$

$$(p(x)u_2)' + [\lambda_2 w(x) - q(x)]\, u_2 = 0. \tag{5}$$

If we multiply the first equation through by u_2 and the second equation through by u_1 then we have,

$$u_2(p(x)u_1)' + u_2\,[\lambda_1 w(x) - q(x)]\, u_1 = 0, \tag{6}$$

$$u_1(p(x)u_2)' + u_1\,[\lambda_2 w(x) - q(x)]\, u_2 = 0. \tag{7}$$

Subtracting these equations from each other gives,

$$u_2(p(x)u_1)' - u_1(p(x)u_2)' + u_2\lambda_1 w(x)u_1 - u_1\lambda_2 w(x)u_2 = 0 \iff \tag{8}$$

$$u_2(p(x)u_1)' - u_1(p(x)u_2)' = [\lambda_2 - \lambda_1]\, w(x)u_1u_2. \tag{9}$$

[1]We now work the following using Newton's notation for derivative. It's easier this way.

Now, here is the first major trick. We must notice that,

$$\frac{d}{dx}\left[p(x)\left(u_2u_1' - u_1u_2'\right)\right] = p'(x)\left[u_2'(x)u_1'(x) + u_2(x)u_1''(x) - u_1'(x)u_2'(x) - u_1(x)u_2''(x)\right] \tag{10}$$

$$= u_2(p(x)u_1)' - u_1(p(x)u_2)', \tag{11}$$

Thus if we integrate both sides of (9) from a to b we have that,

$$\int_a^b u_2(p(x)u_1)' - u_1(p(x)u_2)'dx = \int_a^b \frac{d}{dx}\left[p(x)\left(u_2u_1' - u_1u_2'\right)\right]dx \tag{12}$$

$$= p(b)\left(u_2(b)u_1'(b) - u_1(b)u_2'(b)\right) - p(a)\left(u_2(a)u_1'(a) - u_1(a)u_2'(a)\right) \tag{13}$$

The second major trick is to show that this quantity is zero because of (2)-(3). A dint of algebra shows that,

$$p(b)\left(u_2(b)u_1'(b) - u_1(b)u_2'(b)\right) = p(b)\left(-\frac{u_2(b)u_1(b)I_1}{I_2} + \frac{I_1u_1(b)u_2(b)}{I_2}\right) \tag{14}$$

$$= 0. \tag{15}$$

A similar statement holds for the lower-bound. Lastly, we note that from the previous steps we have that,

$$[\lambda_2 - \lambda_1]\int_a^b w(x)u_1(x)u_2(x)dx = 0. \tag{16}$$

Since $\lambda_1 \neq \lambda_2$ we conclude that,

$$\int_a^b w(x)u_1(x)u_2(x)dx = 0, \tag{17}$$

which implies that eigenfunctions associated with different eigenvalues are orthogonal with respect to the inner-product,

$$\langle f, g\rangle = \int_a^b f(x)g(x)w(x)dx. \tag{18}$$

We have seen this result before. Recall that in homework 3 we studied this equation for $p(x) = w(x) = 1$ and $q(x) = 0$ and found cosine solutions that were orthogonal with respect to the same inner-product. This orthogonal structure is quite general and the Fourier series is just one manifestation. There are infinitely many others! Some are useful but some are not.

1.2. **Bessel's Equation.** Show that if $p(x) = x$, $q(x) = \nu^2/x$ and $w(x) = x/\lambda$ then (1) becomes $x^2u'' + xu' + (x^2 - \nu^2)u = 0$, which is known as Bessel's equation of order ν.

We have from (1) that,

$$Lu = \frac{\lambda}{x}\left(-\frac{d}{dx}\left[x\frac{du}{dx}\right] + \frac{\nu^2}{x}u\right) \tag{19}$$

$$= -\frac{\lambda}{x}\left(xu'' + u' - \frac{\nu^2}{x}u\right) = \lambda u \tag{20}$$

$$\iff xu'' + u' + \left(x - \frac{\nu^2}{x}\right)u = 0 \tag{21}$$

$$\iff x^2u'' + xu' + \left(x^2 - \nu^2\right)u = 0, \tag{22}$$

which agrees with the given equation. See EK.5.5pg198 for an investigation of this equation.

1.3. **Fourier Bessel Series.** A solution to Bessel's equation is for $\nu = n \in \mathbb{N}$,

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n}m!(n+m)!}, \quad n = 1, 2, 3, \ldots \tag{23}$$

which is called Bessel's function of the first-kind of order n. Since these functions manifest from a SL problem they naturally orthogonal and have an orthogonality condition,

$$\langle J_n(xk_{n,m}), J_n(xk_{n,i})\rangle = \int_0^R xJ_n(xk_{n,m})J_n(xk_{n,i})dx = \frac{\delta_{mi}}{2}\left[RJ_{n+1}(k_{nm}R)\right]^2. \tag{24}$$

Using this show that the coefficients in the Fourier-Bessel series,

$$f(x) = \sum_{m=1}^{\infty} a_m J_n(k_{n,m}x), \tag{25}$$

are given by,

$$a_i = \frac{2}{R^2 J_{n+1}^2(k_{n,i}R)} \int_0^R x J_n(k_{n,i}R) f(x) dx, \quad i = 1, 2, 3, \dots \tag{26}$$

We will explicitly find these solutions in class but the practical matter is how to use them. If someone provides the functions and orthogonality condition then we can cavalierly expand in this basis. [2] Before we do this we should notice,

- The relation (24) implies that the orthogonality is on the argument not the specific Bessel function in question. That is, we have infinitely many Bessel functions forming a class indexed by the integer n and the functions in the class are mutually orthogonal. This is seen by the the fact that the Kronecker delta function has index (m, i) and not (m, n).

We now proceed, as usual, by considering an appropriately chosen inner-product on (25).[3] Doing so gives,

$$\langle J_n(k_{m,i}x), f(x)\rangle = \left\langle J_i(k_{m,i}x), \sum_{m=1}^{\infty} a_m J_n(k_{n,m}x) \right\rangle \tag{27}$$

$$= \sum_{m=1}^{\infty} a_m \langle J_n(k_{m,i}x), J_n(k_{n,m}x)\rangle \tag{28}$$

$$= \sum_{m=1}^{\infty} a_m \frac{\delta_{mi}}{2} \left[R J_{n+1}(k_{nm}R)\right]^2 \tag{29}$$

$$= a_i \frac{\delta_{ii}}{2} \left[R J_{n+1}(k_{ni}R)\right]^2, \tag{30}$$

which implies that,

$$a_i = \frac{2}{\left[R J_{n+1}(k_{ni}R)\right]^2} \langle J_n(k_{m,i}x), f(x)\rangle \tag{31}$$

$$= \frac{2}{\left[R J_{n+1}(k_{n,i}R)\right]^2} \int_0^R x f(x) J_n(k_{m,i}x) dx. \tag{32}$$

2. Power-Series Solutions to ODE's and Hyperbolic Trigonometric Functions

Consider the ordinary differential equation:

$$y'' - y = 0 \tag{33}$$

2.1. General Solution - Standard Form.

Show that the solution to (33) is given by $y(x) = c_1 e^x + c_2 e^{-x}$.

You could derive this solution from (33) but the beauty of a differential equation is that you can always check.

$$y'' - y = c_1 e^x + (-1)^2 c_2 e^{-x} - c_1 e^x + c_2 e^{-x} = 0. \tag{34}$$

2.2. General Solution - Nonstandard Form.

Show that $y(x) = b_1 \sinh(x) + b_2 \cosh(x)$ is a solution to (33) where $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$ and $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$.

First we notice the derivative relations on the hyperbolic trigonometric functions,

$$\begin{aligned} y(x) &= b_1 \sinh(x) + b_2 \cosh(x), \\ y'(x) &= b_1 \cosh(x) + b_2 \sinh(x), \\ y''(x) &= b_1 \sinh() + b_2 \cosh(x), \end{aligned}$$

which then gives,

$$y'' - y = b_1 \sinh(x) + b_2 \cosh(x) - (b_1 \sinh(x) + b_2 \cosh(x)) = 0, \tag{35}$$

and shows that this is also a solution to (33).

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[2] You should remember that you could owe some mathematician money at this point but hey a trade is a trade.

[3] See HW2.5.4 for a reminder.

2.3. **Conversion from Standard to Nonstandard Form.** Show that if $c_1 = \dfrac{b_1+b_2}{2}$ and $c_2 = \dfrac{b_1-b_2}{2}$ then $y(x) = c_1e^x + c_2e^{-x} = b_1\cosh(x) + b_2\sinh(x)$.

Direct substitution gives,

$$\begin{aligned} y(x) &= c_1e^x + c_2e^{-x}, \\ &= \left(\frac{b_1+b_2}{2}\right)e^x + \left(\frac{b_1-b_2}{2}\right)e^{-x}, \\ &= \frac{b_1(e^x+e^{-x})}{2} + \frac{b_2(e^x-e^{-x})}{2}, \\ &= b_1\cosh(x) + b_2\sinh(x), \end{aligned}$$

which shows the two solutions are equivalent.

2.4. **Relation to Power-Series.** Assume that $y(x) = \sum_{n=0}^{\infty} a_nx^n$ to find the general solution of (33) in terms of the hyperbolic sine and cosine functions. [4]

Assume that $y(x) = \sum_{n-0}^{\infty} a_nx^n$ and find the general solution of (33). First, we have the following derivative relations.

(37) $$y(x) = \sum_{n=0}^{\infty} a_nx^n$$

(38) $$y'(x) = \sum_{n=0}^{\infty} a_n(n)x^{n-1}$$

(39) $$y''(x) = \sum_{n=0}^{\infty} a_n(n)(n-1)x^{n-2}.$$

Thus the ODE is now,

(40) $$y'' - y = \sum_{n=0}^{\infty} a_n(n)(n-1)x^{n-2} - \sum_{n=0}^{\infty} a_nx^n$$

(41) $$= \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k - \sum_{k=0}^{\infty} a_kx^k$$

(42) $$= \sum_{k=0}^{\infty} [a_{k+2}(k+2)(k+1) - a_k]x^k = 0.$$

The only polynomial, which is zero regardless of its variable is the zero-function. This defines the recurrence relation,

(43) $$[a_{k+2}(k+2)(k+1) - a_k] = 0, \text{ for } k = 1, 2, 3, \ldots.$$

We now write down a_{k+2} in terms of a_k,

(44) $$a_{k+2} = \frac{a_k}{(k+2)(k+1)}, \text{ for } k = 1, 2, 3, \ldots,$$

and try to find a pattern by choosing various k. Since the recurrence relation relates a coefficient to the coefficient two away it splits the coefficients into even and odd patterns.

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[4]The hyperbolic sine and cosine have the following Taylor's series representations centred about $x = 0$,

(36) $$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \qquad \sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

It is worth noting that these are basically the same Taylor series as cosine/sine with the exception that the signs of the terms do not alternate. From this we can gather a final connection for wrapping all of these functions together. If you have the Taylor series for the exponential function and extract the even terms from it then you have the hyperbolic cosine function. Taking the hyperbolic cosine function and alternating the sign of its terms gives you the cosine function. Extracting the odd terms from the exponential function gives the same statements for the hyperbolic sine and sine functions. The reason these functions are connected via the imaginary number system is because when i is raised to integer powers it will produce these exact sign alternations. So, if you remember $e^x = \sum_{n=0}^{\infty} x^n/n!$ and $i = \sqrt{-1}$ then the rest (hyperbolic and non-hyperbolic trigonometric functions) follows!

Even Coefficients	Odd Coefficients
$a_2 = \dfrac{a_0}{2 \cdot 1}$	$a_3 = \dfrac{a_1}{3 \cdot 2 \cdot 1}$
$a_4 = \dfrac{a_4}{4 \cdot 3} = \dfrac{a_0}{4 \cdot 3 \cdot 2 \cdot 1}$	$a_5 = \dfrac{a_3}{5 \cdot 4} = \dfrac{a_1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$
$\vdots$	$\vdots$
$a_{2k} = \dfrac{a_0}{(2k)!}$	$a_{2k+1} = \dfrac{a_1}{(2k+1)!}$

The last entries are the generalization of the pattern where the following are noticed:[5]

(1) The even coefficients all re-curse back to a_0 and the odd coefficients recurse back to a_1. Nothing more can be said about these two coefficients.
(2) As the recursion is applied a product forms in the denominator. This product is precisely the factorial of the coefficient's subscript.

Using this information we now have the following,

$$y(t) = \sum_{n=0}^{\infty} a_n x^n \tag{45}$$

$$= \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1} \tag{46}$$

$$= a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \tag{47}$$

$$= a_0 \cosh(x) + a_1 \sinh(x), \tag{48}$$

3. Conservation Laws in One-Dimension

Recall that the conservation law encountered during the derivation of the heat equation was given by,

$$\frac{\partial u}{\partial t} = -\kappa \nabla \phi, \tag{49}$$

which reduces to

$$\frac{\partial u}{\partial t} = -\kappa \frac{\partial \phi}{\partial x}, \quad \kappa \in \mathbb{R} \tag{50}$$

in one-dimension of space.[6] In general, if the function $u = u(x,t)$ represents the density of a physical quantity then the function $\phi = \phi(x,t)$ represents its flux. If we assume the ϕ is proportional to the negative gradient of u then, from (50), we get the one-dimensional heat/diffusion equation (**??**).[7]

3.1. **Transport Equation.** Assume that ϕ is proportional to u to derive, from (50), the convection/transport equation, $u_t + cu_x = 0 \;\; c \in \mathbb{R}$.

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[5]A generalized pattern is not always possible to find. However, it should be noted that once the recursion relation is found the job is done. Every coefficient can be known in terms of ones that precede connecting coefficients to a_0 and/or a_1. These originators are actually the same as the unknown constants c_1 and c_2 found in the homogeneous solution and can be found via initial conditions. Thus, initial conditions will allow the evaluation of the infinite series to arbitrary decimal precision, which is good enough for tunnel work, as they say.

[6]When discussing heat transfer this is known as Fourier's Law of Cooling. In problems of steady-state linear diffusion this would be called Fick's First Law. In discussing electricity u could be charge density and q would be its flux.

[7]AKA Fick's Second Law associated with linear non-steady-state diffusion.

Assume that ϕ is proportional to u, to derive the convection/transport equation $u_t + cu_x = 0$

$$\begin{aligned}
\phi &= \alpha u \\
\frac{\partial \phi}{\partial x} &= \alpha \frac{\partial u}{\partial x} \\
\frac{\partial u}{\partial t} &= -k\frac{\partial Q}{\partial x} \Rightarrow \frac{\partial u}{\partial t} = -\alpha k \frac{\partial u}{\partial x} \Rightarrow u_t + cu_x = 0
\end{aligned}$$

3.2. **General Solution to the Transport Equation.** Show that $u(x,t) = f(x-ct)$ is a solution to this PDE.

From $u(x,t) = f(x-ct)$ we have that, $u_t = -cf'(x)$ and $u_x = f'(x)$ which gives

(51)
$$u_t + cu_x = -cf'(x) + cf'(x) = 0.$$

Thus the travelling wave $f(x-ct)$ is a general solution to the transport equation.

3.3. **Diffusion-Transport Equation.** If both diffusion and convection are present in the physical system then the flux is given by, $\phi(x,t) = cu - du_x$, where $c, d \in \mathbb{R}^+$. Derive from, (50), the convection-diffusion equation $u_t + cu_x - du_{xx} = 0$.

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= c\frac{\partial u}{\partial x} - d\frac{\partial^2 u}{\partial x^2} \\
\frac{\partial u}{\partial t} &= -k\frac{\partial \phi}{\partial x} \Rightarrow \frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x} + d\frac{\partial^2 u}{\partial x^2} \\
&\Rightarrow u_t + cu_x - du_{xx} = 0
\end{aligned}$$

3.4. **Convection-Diffusion-Decay.** If there is also energy/particle loss proportional to the amount present then we introduce to the convection-diffusion equation the term λu to get the convection-diffusion-decay equation,[8]

For $u_t = -\lambda u$ where $\lambda \geq 0$ we have decay. Including this additively in the previous equation gives, $u_t = du_{xx} - cu_x - \lambda u = 0$, which is the convection-diffusion-decay equation.

3.5. **General Importance of Heat/Diffusion Problems.** Given that,

(52)
$$u_t = Du_{xx} - cu_x - \lambda u.$$

Show that by assuming, $u(x,t) = w(x,t)e^{\alpha x - \beta t}$, (52) can be transformed into a heat equation on the new variable w where $\alpha = c/(2D)$ and $\beta = \lambda + c^2/(4D)$.[9]

Assume that $u(x,t) = w(x,t)e^{\alpha x - \beta t}$ and show that (a) can be transformed into a heat equation on the variable w where $\alpha = \frac{c}{2D}$ and $\beta = \lambda + \frac{c^2}{4D}$

$$\begin{aligned}
u_t &= w_t e^{\alpha x - \beta t} + w\beta e^{\alpha x - \beta t} \\
u_x &= w_x e^{\alpha x - \beta t} + w\alpha e^{\alpha x - \beta t} \\
u_{xx} &= w_{xx} e^{\alpha x - \beta t} + 2w_x \alpha e^{\alpha x - \beta t} + w{\alpha}^2 e^{\alpha x - \beta t} \\
u_t &= Du_{xx} - cu_x - \lambda u \\
w_t e^{\alpha x - \beta t} - w\beta e^{\alpha x - \beta t} &= Dw_{xx} e^{\alpha x - \beta t} + D2w_x \alpha e^{\alpha x - \beta t} + Dw{\alpha}^2 e^{\alpha x - \beta t} - \\
&\quad -cw_x e^{\alpha x - \beta t} - cw\alpha e^{\alpha x - \beta t} - \lambda w e^{\alpha x - \beta t} \\
&\Rightarrow w_t - \beta w = Dw_{xx} + 2D\alpha w_x + Dw{\alpha}^2 - cw_x - c\alpha w - \lambda w \\
w_t &= Dw_{xx} + (2D\alpha - c)w_x + (\beta - c\alpha + D{\alpha}^2 - \lambda)w \\
w_t &= Dw_{xx} + 2D\left(\frac{c}{2D} - c\right)w_x + \left(\lambda + \frac{c^2}{4D} - \frac{c^2}{2D} + \frac{Dc^2}{4D^2}\right)w \\
w_t &= Dw_{xx} \leftarrow \text{heat equation on variable } w
\end{aligned}$$

[8]The u_{xx} term models diffusion of energy/particles while u_x models convection, u models energy/particle loss/decay. The final term should not be surprising! Wasn't the appropriate model for radioactive/exponential decay $Y' = -\alpha^2 Y$?

[9]This shows that the general PDE (52) can be solved using heat equation techniques.

Show that the following functions are solutions to their corresponding PDE's.

4.1. Right and Left Travelling Wave Solutions. $u(x,t) = f(x-ct) + g(x+ct)$ for the 1-D wave equation.

First we note the derivative relations.

$$\frac{\partial u}{\partial t} = -cf' + cg',$$
$$\frac{\partial^2 u}{\partial t^2} = c^2 f'' + c^2 g'',$$
$$\frac{\partial u}{\partial x} = f' + g',$$
$$\frac{\partial^2 u}{\partial x^2} = f'' + g''.$$

Thus,

$$u_{tt} - c^2 u_{xx} = c^2(f'' + g'') - c^2(f'' + g'') = 0, \tag{53}$$

implies that the linear combination of right and left traveling waves is a general solution to the 1-D wave equation. An importance consequence of this fact is that phenomenon modeled by this equation must have a finite-speed of propagation. This may seem like common sense however, there are phenomenon modeled by equations which propagate data at infinite speeds! The heat equation is one such example.

4.2. Decaying Fourier Mode. $u(x,t) = e^{-4\omega^2 t}\sin(\omega x)$ where $c = 2$ for the 1-D heat equation.

$$u_t - c^2 u_{xx} = -4\omega^2 e^{-4\omega^2 t}\sin(\omega x) - c^2\omega^2 e^{-4\omega^2 t}\sin(\omega x) = 0, \quad \text{for } c = 2. \tag{54}$$

Notice that since ω can be anything then a good choice might be $\omega = n\pi/L$. Also, since this equation is linear the linear combination of solutions is also a solution. Thus, time decaying Fourier modes ought to solve this problem......

4.3. Radius Reciprocation. $u(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$ for the 3-D Laplace equation.

The solution is symmetric under variable interchange. Thus,

$$\frac{\partial u}{\partial x} = \frac{-x}{(x^2+y^2+z^2)^{3/2}} \tag{55}$$
$$\frac{\partial^2 u}{\partial x^2} = \frac{3x^2}{(x^2+y^2+z^2)^{5/2}} + \frac{-1}{(x^2+y^2+z^2)^{3/2}} \tag{56}$$
$$\frac{\partial^2 u}{\partial y^2} = \frac{3y^2}{(x^2+y^2+z^2)^{5/2}} + \frac{-1}{(x^2+y^2+z^2)^{3/2}} \tag{57}$$
$$\frac{\partial^2 u}{\partial z^2} = \frac{3z^2}{(x^2+y^2+z^2)^{5/2}} + \frac{-1}{(x^2+y^2+z^2)^{3/2}} \tag{58}$$

Thus,

$$u_{xx} + u_{yy} + u_{zz} = \frac{3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} - \frac{3}{(x^2+y^2+z^2)^{3/2}} = 0 \tag{59}$$

This result tells us that in three-space a solution to Laplace's equation, an equation that models potential fields, is the function $u = 1/r$.

4.4. Driving/Forcing Affects. $u(x,y) = x^4 + y^4$ where $f(x,y) = 12(x^2+y^2)$ for the 2-D Poisson equation.

First we have,

$$\frac{\partial u}{\partial x} = 4x^3, \tag{60}$$
$$\frac{\partial^2 u}{\partial y^2} = 12y^2, \tag{61}$$
$$\frac{\partial u}{\partial y} = 4y^3, \tag{62}$$
$$\frac{\partial^2 u}{\partial x^2} = 12x^2. \tag{63}$$

Thus,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 12x^2 + 12y^2 = f(x, y). \tag{64}$$

Note: The PDE in question are,

- Laplace's equation : $\triangle u = 0$
- Poisson's equation : $\triangle u = f(x, y, z)$
- Heat/Diffusion Equation : $u_t = c^2 \triangle u$
- Wave Equation : $u_{tt} = c^2 \triangle u$

and can be found on page 563 of Kryszig. The following will outline some common notations. It is assumed all differential operators are being expressed in Cartesian coordinates.[10]

- Notations for partial derivatives,

$$\frac{\partial u}{\partial x} = u_x = \partial_x u \tag{65}$$

- Nabla the differential operator,

$$\nabla = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \tag{66}$$

- Gradient of a scalar function,

$$\nabla u = \begin{bmatrix} \partial_x u \\ \partial_y u \\ \partial_z u \end{bmatrix} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \tag{67}$$

- Divergence of a vector,

$$\nabla \cdot \boldsymbol{v} = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \partial_x v_1 + \partial_y v_2 + \partial_z v_3 \tag{68}$$

- Curl of a vector,

$$\nabla \times \mathbf{v} = \begin{bmatrix} \partial_y v_3 - \partial_z v_2 \\ \partial_z v_1 - \partial_x v_3 \\ \partial_x v_2 - \partial_y v_1 \end{bmatrix} \tag{69}$$

- Notations for the Laplacian,

$$\triangle u = \nabla \cdot \nabla u = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \cdot \begin{bmatrix} \partial_x u \\ \partial_y u \\ \partial_z u \end{bmatrix} \tag{70}$$

$$= \partial_{xx} u + \partial_{yy} u + \partial_{zz} u \tag{71}$$

$$= u_{xx} + u_{yy} + u_{zz} \tag{72}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \tag{73}$$

[10]Of course others have worked out the common coordinate systems, which requires some elbow grease and the multivariate chain rule. Those interested in the results can find them at Nabla in Cylindrical and Spherical