Linear Algebra Overview

or How I Learned to Stop Worrying and Love the Linear Transformation

Scott Strong

June 28, 2010

Outline

Outline

Introduction

Motivational Peptalk

Robert McKee on Storytelling

Storytelling is the most powerful way to put ideas into the world today.

Robert Moss : Dreamgates

Australian Aborigines say that the big stories—the stories worth telling and retelling, the ones in which you may find the meaning of your life—are forever stalking the right teller, sniffing and tracking like predators hunting their prey in the bush.

Introduction

Motivational Peptalk

Barry Lopez : Crow and Weasel

If stories come to you, care for them. And learn to give them away where they are needed. Sometimes a person needs a story more than food to stay alive.

Gilda Radner

I wanted a perfect ending. Now I've learned, the hard way, that some poems don't rhyme, and some stories don't have a clear beginning, middle, and end. Life is about not knowing, having to change, taking the moment and making the best of it, without knowing what's going to happen next. Delicious ambiguity.

Take Home Message

The Singular Value Decomposition

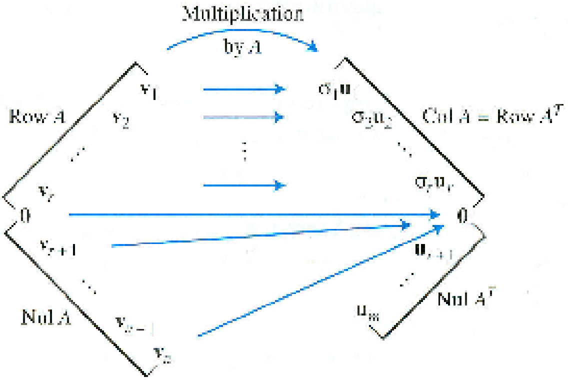

FIGURE 4 The four fundamental subspaces and the action of A.

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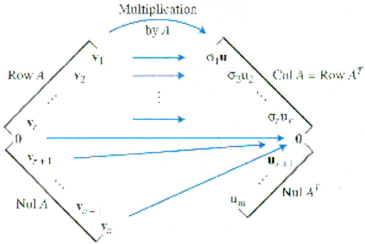


FIGURE 4 The four fundamental subspaces and the action of A.

Linear Transformations of $\mathbb{R}^n$

Every linear transformation, **A**, of finite-dimensional space, can be decomposed into the product of transformations $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$ where Σ characterizes the invertibility of the mapping **A** and **V** and **U** act as geometry preserving isometries.

Outline

Functions/Mappings/Transformations

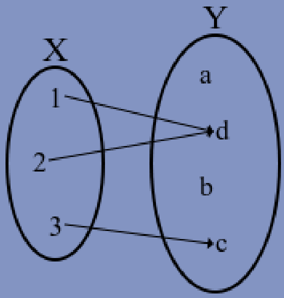
how to get from here to there

Definition: Function

Wikipedia: Function - Precises Definition

Working Definition: A *function*, f, is a rule that uniquely maps elements in its *domain*, D, to elements in its *range*, R. We write, $f : D \to R$ and call the set of ordered pairs $(x, f(x))$ the *graph* of f.

Diagram: Total Function



Example of Total Function ($D = X$)

- Let $f(x) = x^2$ where $D = X = \mathbb{R}$ but $R \neq Y = \mathbb{R}$. We say f is total in X but not onto Y.

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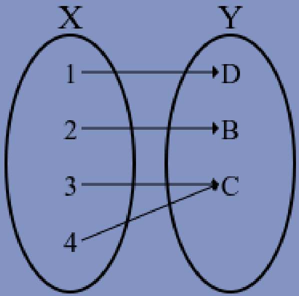
Diagram: Surjection (Onto)



Example of Surjective (Onto) function

- Let $f(x) = x^2$ where $D = X = \mathbb{R}$ and $R = Y = [0, \infty)$.

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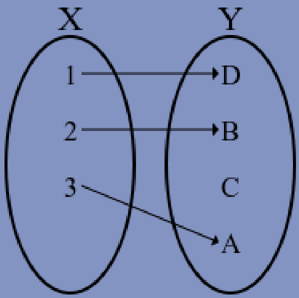
Diagram: Injection (1-to-1)



Example of Injective (One-to-One) function

- Let $f(x) = x^2$ where $D = X = [0, \infty)$ is mapped one-to-one into $Y = \mathbb{R}$ and onto $R = [0, \infty)$

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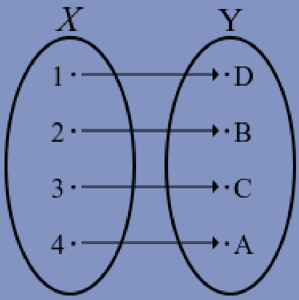
Diagram: Bijection



Example of Bijective (one-to-one and onto) function

- Let $f(x) = x^2$, $D = X = [0, \infty)$ and $R = Y = [0, \infty)$. The mapping $f : [0, \infty) \to [0, \infty)$ is a bijection.

Outline

Linear Mappings

Part I : General Story

Definition: Linear Map

The transformation $T : X \to Y$ is linear if,

$$T(x_1 + x_2) = T(x_1) + T(x_2),$$
$$T(\alpha x_1) = \alpha T(x_1),$$

for any $x_1, x_2 \in X$ and any $\alpha \in \mathbb{R}$. Wikipedia : Linear Map

Consequences

- Fixed Identity: $T(0) = 0$
- Linear Combinations:
$$T\left(\sum_{j=1}^{N} \alpha_j x_j\right) = \sum_{j=1}^{N} \alpha_j T(x_j)$$

Examples

- Derivative/Integral
- Linear Homogeneous: $f(x) = ax,\ a \in \mathbb{R}$

Linear Mappings

Part II : Matrix Transformations

General Linear Equation

Suppose $f : \mathbb{R}^n \to \mathbb{R}$. If f is linear then it must take the form $f(x_1, x_2, x_3, \ldots, x_n) = f(\mathbf{x}) = a_j x_j$ for $a_j \in \mathbb{R}$. If $n = 3$ then we have the equation of a plane, $f(\mathbf{x}) = a_1 x_1 + a_2 x_2 + a_3 x_3$.

General Linear System

Suppose we $f_i : \mathbb{R}^n \to \mathbb{R}$ where $i = 1, 2, 3, \ldots, m$. If each f_i is linear then it must take the form $f_i(\mathbf{x}) = a_{ij} x_j$. Collectively, this is called a system of linear equations. For $n = 2$ and $m = 3$ we have _.

Matrix Transformation

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix}$$

$\mathbf{A}(\mathbf{x}) = \mathbf{f}$ is a Linear Mapping

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Matrix Fundamentals

Equivalent Forms for Linear Systems

Linear System of Equations: m-many linear objects in n-dimensions

$$\begin{array}{cccccc} a_{11}x_1+ & a_{12}x_2+ & a_{13}x_3+ & \cdots & +a_{1n}x_n & = f_1, \\ a_{21}x_1+ & a_{22}x_2+ & a_{23}x_3+ & \cdots & +a_{2n}x_n & = f_2, \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ a_{m1}x_1+ & a_{m2}x_2+ & a_{m3}x_3+ & \cdots & +a_{mn}x_n & = f_m, \end{array}$$

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Linear Combinations of Vectors: Alternate Representations of $\mathbf{f}$

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ \vdots \\ a_{m3} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix}$$

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Matrix Transformations: $\mathbf{Ax} = \mathbf{f}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & \ddots & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix}$$

Summary

Linear Mappings and the Fundamental Question

Matrix Transformations

A transformation of *n*-dimensional space is defined as,

$$\mathbf{A} : \mathbb{R}^n \rightarrow \mathbb{R}^m, \quad n, m \in \mathbb{N}. \tag{1}$$

If the transformation is linear then $\mathbf{A}$ can be represented as a matrix, which acts on vectors from $\mathbb{R}^n$ and returns vectors from $\mathbb{R}^m$. Symbolically, we have $\mathbf{A}(\mathbf{x}) = \mathbf{A}\mathbf{x} = \mathbf{f} \in \mathbb{R}^m$ for $\mathbf{x} \in \mathbb{R}^n$.

Fundamental Question

Given a linear transformation, $\mathbf{A}$, of $\mathbb{R}^n$ what can be said about its inverse transformation $\mathbf{A}^{-1}$ of $\mathbb{R}^m$?

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Singular Value Decomposition (SVD)

Linear Transformations of $\mathbb{R}^n$ are well-understood.

Theorem: SVD

Let $\mathbf{A}$ be linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. The matrix representation of $\mathbf{A}$ has the following decomposition,

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}}, \tag{2}$$

where $\mathbf{V} : \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{U} : \mathbb{R}^m \to \mathbb{R}^m$ are orthogonal matricies and $\boldsymbol{\Sigma} : \mathbb{R}^n \to \mathbb{R}^m$ is diagonal.

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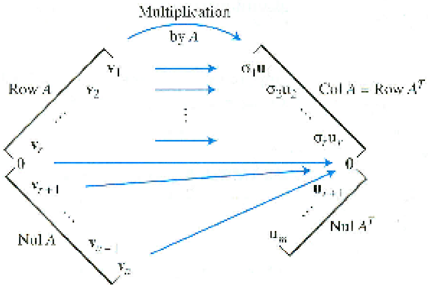

FIGURE 4 The four fundamental subspaces and the action of A.

SVD Breakdown - Part I

Action of the Singular Value Matrix

A Diagonal Problem

For every $\mathbf{A}$ we now have,

$$\mathbf{A}\mathbf{x} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}\mathbf{x} = \mathbf{U}\Sigma\tilde{\mathbf{x}} = \mathbf{y},$$

where $\tilde{\mathbf{x}} = \mathbf{V}^{\mathrm{T}}\mathbf{x}$. Similarly,

$$\Sigma\tilde{\mathbf{x}} = \mathbf{U}^{\mathrm{T}}\mathbf{y} = \tilde{\mathbf{y}}.$$

Roughly, this means that when $\sigma_i \neq 0$ we have,

$$\Sigma\tilde{\mathbf{x}} = \tilde{\mathbf{y}} \iff \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & & \ddots & \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \vdots \end{bmatrix} \implies \tilde{x}_i = \frac{\tilde{y}_i}{\sigma_i}.$$

SVD Breakdown - Part II

Action of **A** as seen through SVD

Orthogonal Transformations

Both **U** and **V** are orthogonal matrices. If **M** is an orthogonal matrix then:

- Angle Preservation: $\mathbf{M}(\mathbf{x}\cdot\mathbf{y}) = \mathbf{x}\cdot\mathbf{y}$
- Length Preservation: $||\mathbf{M}\mathbf{x}|| = ||\mathbf{x}||$
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Key Point

- The action of $\mathbf{V}^{\mathrm{T}}$ preserves the geometry of the input-vector space.
- The action of $\mathbf{U}^{\mathrm{T}}$ preserves the geometry of the output vector-space.

Action of A by Σ

- $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^m$ reduces to $\Sigma : \tilde{\mathbb{R}}^n \to \tilde{\mathbb{R}}^m$
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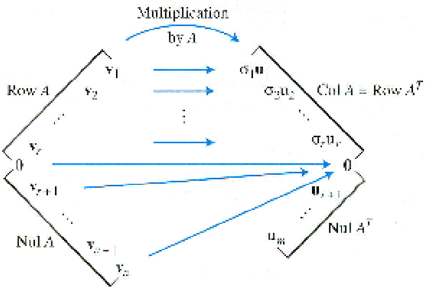
SVD Breakdown - Part III

Morse-Penrose Inverse



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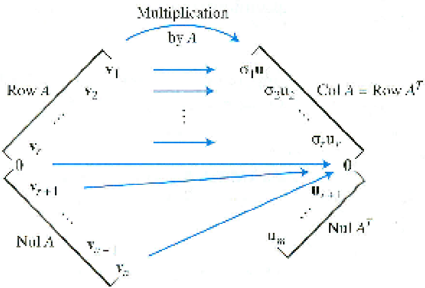
Back to the Problem at Hand: Invertible Linear Mappings

Clearly, $\mathbf{A}$ cannot always be one-to-one and onto (bijective). However, we can *restrict* $\mathbf{A}$ to a domain and range for which it is bijective.

- For such a restriction we have $\mathbf{A} = \mathbf{U}_r\Sigma_r\mathbf{V}_r^{\mathrm{T}}$, which gives rise to a pseduoinverse or the Morse-Penrose Inverse $\mathbf{A}^+ = \mathbf{V}_r\Sigma^{-1}\mathbf{U}_r^{\mathrm{T}}$.

SVD Breakdown - Part III

Morse-Penrose Inverse



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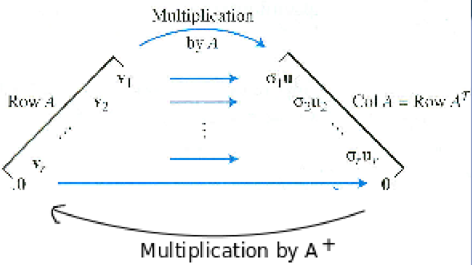
Clearly, **A** cannot always be one-to-one and onto (bijective). However, we can *restrict* **A** to a domain and range for which it is bijective.

- For such a restriction we have $\mathbf{A} = \mathbf{U}_r \Sigma_r \mathbf{V}_r^{\mathrm{T}}$, which gives rise to a pseudoinverse or the Morse-Penrose Inverse $\mathbf{A}^+ = \mathbf{V}_r \Sigma^{-1} \mathbf{U}_r^{\mathrm{T}}$.

SVD Breakdown - Part III

Morse-Penrose Inverse



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Summary

Characterization of Linear Transformations on $\mathbb{R}^n$

For every linear transformation $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^m$

Wikipedia: SVD — $\mathbf{A}$ admits the singular value decomposition $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$

Wikipedia: Unitary Matrix — $\mathbf{U}$ and $\mathbf{V}^{\mathrm{T}}$ are orthogonal matrices

Wikipedia: Isometry — $\mathbf{U}$ and $\mathbf{V}^{\mathrm{T}}$ are isometries of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively.

Wikipedia: Diagonal Matrix — Σ is diagonal and reduces the linear system to a diagonal problem.

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Outline