

The skin effect

calculate \vec{E} in a conducting wire for AC current.
start with wave eqn:

$$\nabla^2 \vec{E} - \frac{4\pi\sigma\mu}{c^2} \frac{\partial \vec{E}}{\partial t} = \frac{\epsilon\mu}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

claim: very good conductor, throw out term on RHS
show - $E \sim e^{-iwt}$

$$\rightarrow \nabla^2 E + \frac{i4\pi\sigma\mu w}{c^2} \vec{E} = -\frac{w^2 \epsilon \mu}{c^2} \vec{E}$$

compare magnitude of the terms

$$4\pi\sigma \text{ vs. } w\epsilon$$

if $w \ll \frac{4\pi\sigma}{\epsilon}$ (either low freq, or high σ limit)

we can drop RHS, leaving a diffusion eqn:

$$\nabla^2 \vec{E} - \frac{4\pi\sigma\mu}{c^2} \frac{\partial \vec{E}}{\partial t} = 0 \quad \begin{matrix} \text{looks like Schrödinger eqn,} \\ \text{but S.E. has } i \text{ in diff eqn} \end{matrix}$$

turn this into an eqn. for $\vec{J} = \sigma \vec{E}$,

$$\text{use } \vec{J} = \vec{J}_0 e^{-iwt}$$

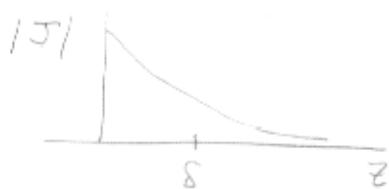
$$\nabla^2 \vec{J}_0 + K^2 \vec{J}_0 = 0 \quad \text{with } K^2 = i \frac{4\pi\sigma\mu w}{c^2}$$

$$\text{since } \sqrt{i} = \frac{1+i}{\sqrt{2}} \quad K = (1+i) \frac{\sqrt{2\pi\sigma\mu w}}{c} = \frac{1+i}{8}$$

$$\delta = \frac{c}{\sqrt{2\pi\sigma\mu w}} = \text{skin depth}$$

in slab geometry, solution is

$$\vec{E} = \frac{\vec{J}_0}{\sigma} = \frac{\vec{J}_0}{\sigma} e^{iKz}$$



with two surfaces, E on both sides:



$$E(z) = E_0 [e^{ik(z-w)} + e^{-ik(z-w)}]$$
$$= 2E_0 \cos(k(z-w))$$

so magnitude is



degree of penetration depends on

skin depth $\delta \propto 1/\sqrt{\omega}$

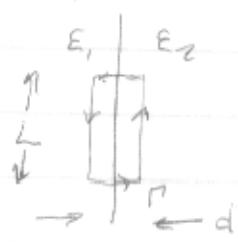
for low freq. δ is large, field is uniform throughout.

Fresnel Equations - reflection + refraction

- Applies to a step-change in refractive index or
- method: take solutions of wave equation in both media
use boundary conditions to match \vec{E}, \vec{H}
express reflected, transmitted amplitudes in terms of incident ampt:
- results: ratios r_s, t_s and r_n, t_n
 r, t depend on polarization direction relative to interface.
 r, t are complex, $R = |r|^2, T = |t|^2$ for power transmission
- Applications
 - method is used in calculating $E(\vec{r})$ in waveguides, resonators, multilayer structures.
 - Fresnel lens are used to design polarization optics, multilayer coatings, interferometers
- Ellipsometry: measure $n(\theta)$ for both polarizations in order to get characteristics of material

Boundary conditions: dielectric interface re HM 1.8

\vec{E} tangential component continuous shorthand: $\vec{E}_x \hat{n}$



$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{1}{c} \frac{\partial}{\partial t} \oint_S \vec{B} \cdot \hat{n} da$$

limit $d \rightarrow 0$

$$(\vec{E}_1 \times \hat{n}_1 - \vec{E}_2 \times \hat{n}_2) L = 0, \vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}$$

\vec{D} normal component continuous



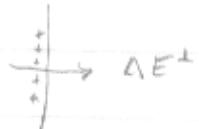
$$\nabla \cdot \vec{D} = 0 \quad (\text{no free charges})$$

$$\oint_S \vec{D} \cdot \hat{n} da = 0 \quad \text{limit } d \rightarrow 0 \\ = (\vec{D}_1 \cdot \hat{n}_1 + \vec{D}_2 \cdot \hat{n}_2) A = 0$$

$$\text{let } \hat{n} = \hat{n}_1 = -\hat{n}_2$$

$$\vec{D}_1 \cdot \hat{n} = \vec{D}_2 \cdot \hat{n} \quad \text{or } \epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}$$

here, jump in E^{\perp} b/c of bound charges.



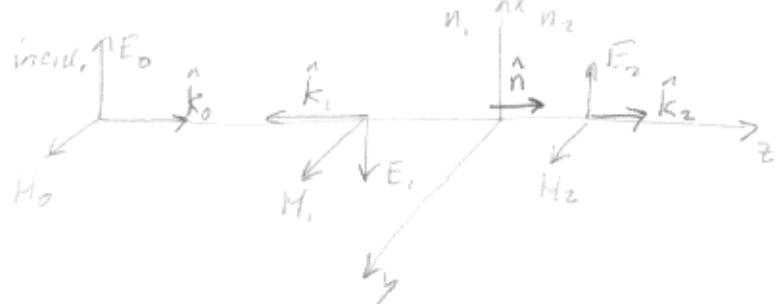
Similarly,

\vec{B} normal comp. contra. (from $\nabla \cdot \vec{B} = 0$)

\vec{H} tang. comp. contra. (no surface currents)

Reflection + transmission : Normal incidence

- 1) define default orientation of the fields



tangential E, H are continuous at interface

Note conventions vary hence sign of n is relative to how fields are set up.

- 2) write wave solutions in different regions

$$\text{incident: } \vec{E}_0 = E_0^ \hat{x} e^{i(k_0 n_1 z - \omega t)}$$

$$\text{refl. } \vec{E}_1 = E_1^ \hat{x} e^{i(-k_0 n_1 z - \omega t)}$$

$$\text{transm. } \vec{E}_2 = E_2^ \hat{x} e^{i(k_0 n_2 z - \omega t)}$$

similar for \vec{H} (call in $+\hat{y}$ direction)

set $t=0$: in this formulation, all is time-independent

- 3) match at boundary $z=0$

$$E_0^ - E_1^ = E_2^$$

$$H_0^ + H_1^ = H_2^$$

$$\text{take } E \rightarrow H \quad \nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \rightarrow \vec{k} \times \vec{E} = \frac{\omega}{c} \mu \vec{H}$$

$$k = k_0 n_i = \frac{\omega}{c} n_i \rightarrow H^ = \frac{n_i}{\mu} E^$$

let $\mu = 1$

$$\rightarrow n_1 (E_0^ + E_1^) = n_2 E_2^$$

- 4) solve for $E_1^$, $E_2^$ in terms of $E_0^$

$$n_1 E_0^ + n_1 E_1^ = n_2 E_0^ - n_2 E_1^$$

$$\text{reflected } E_1^ = E_0^ \frac{(n_2 - n_1)}{n_1 + n_2} \equiv r E_0^$$

$$\text{transm. } E_2^ = E_0^ - E_0^ \frac{n_2 - n_1}{n_1 + n_2} = \frac{2 n_1}{n_1 + n_2} E_0^ \equiv t E_0^$$

↳ ampl. reflection coeff.

Comments:

- for $n_2 > n_1$, e.g. air to glass : $r > 0$ but E changes sign on reflection
- for $n_2 < n_1$, $r < 0$ i.e. no sign change in E

At normal incidence, r, t are real

Power & Intensity reflection coefficients.

$$R = \frac{\langle \vec{S}_1 \rangle \cdot (-\hat{n})}{\langle \vec{S}_0 \rangle \cdot \hat{n}} = \frac{\frac{\epsilon}{8\pi} \operatorname{Re}(\vec{E}_1 \times \vec{H}_1^*) \cdot (-\hat{n})}{\frac{\epsilon}{8\pi} \operatorname{Re}(\vec{E}_0 \times \vec{H}_0^*) \cdot \hat{n}} \quad \text{with } H_0^* = n_1 E_0^*$$

$$= \frac{n_1 |E_1|^2}{n_1 |E_0|^2} = |r|^2 = \left(\frac{n_2 - n_1}{n_2 + n_1} \right)^2$$

$$T = \frac{n_2}{n_1} \frac{|E_2|^2}{|E_1|^2} = \frac{n_2}{n_1} |t|^2 = \frac{4n_1 n_2}{(n_1 + n_2)^2}$$

$$R + T = 1 \quad (\text{note } r^2 + t^2 \neq 1 !)$$

the extra factor of $\frac{n_2}{n_1}$ comes from the fact
that the intensity $|t|^2$ is higher in a medium
with $n > 1$

$$I = V_{ph} \cdot \langle U \rangle = \frac{c}{n} \cdot \epsilon \frac{|E|^2}{8\pi} = \frac{1}{8\pi} n c |E|^2$$

$U \uparrow$ by n^2

$V_{ph} \downarrow$ by $1/n$