MATH 348 - Advanced Engineering Mathematics October 31, 2009

Homework 8, Fall 2009 **Due**: November 26, 2009

Fourier Transforms

1. If a function is not periodic nor can it be periodically extended then the function has no Fourier series representation. However, in this case the function can have a Fourier integral representation. Analysis of the Fourier integral representation[1] reveals the complex Fourier transform pairs:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega, \quad \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \tag{1}$$

We can connect this transform pair to the complex Fourier series. If a function is periodic then there exists a representation of the function in the countably-infinite basis of imaginary-exponential functions. In this case the coefficients[2] of this expansion are given by an integral whose limits of integration are bounded.[3] These coefficients quantify the amplitude of oscillation for each *discrete* frequency of oscillation.[4] In the case of $f(x)$, as above, we have that the function f has a representation in the uncountably-infinite basis of imaginary-exponential functions. In this case the coefficients are given by the previous integral whose limits of integration are unbounded.[5] These coefficients quantify the amplitude of oscillation for each *continuous* frequency of oscillation.

Now, we want to connect all of this to sections 11.7 and 11.8 in our text. We have that the Fourier integral represents functions in the sine/cosine basis without requiring the function to be periodic. Now, what role does symmetry play in this representation? With minimal work we see that if a function is even/odd then the Fourier integral reduces to a Fourier cosine/sine integral.[6] As with our original derivation of transform, if we look at the interplay between the coefficient functions $A(\omega)/B(\omega)$ and $f(x)$ we find the transform pairs:

$$f_c(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}(\omega)\cos(\omega x) d\omega \qquad \hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\cos(\omega x) dx \tag{2}$$

$$f_s(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}(\omega)\sin(\omega x) d\omega \qquad \hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\sin(\omega x) dx \tag{3}$$

We call (2) the Fourier cosine transform pair and, surprisingly, we call (3) the Fourier sine transform.

(a) Show that $f_c(x)$ and $\hat{f}_c(\omega)$ are even functions and that $f_s(x)$ and $\hat{f}_s(\omega)$ are odd functions.[7]

(b) Show that if we assume that $f(x)$ is an even function then (1) defines the transform pair given by (2). Also, show that if $f(x)$ is an odd function then (1) defines the transform pair given by (3).[8]

Given,

$$f(x) = \begin{cases} A, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}, \qquad A, a \in \mathbb{R}^+. \tag{4}$$

(c) On the same graph plot the even and odd extensions of f.

(d) Find the Fourier cosine and sine transforms of f.

(e) Using the Fourier cosine transform show that $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(\pi\omega)}{\pi\omega} d\omega = 1$.

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[1] See classnotes or Kreyszig pgs. 518-519

[2] also called weights

[3] Recall that our derivation lead to $c_n = \frac{1}{2\pi}\int_{-L}^{L} f(x) e^{i\omega_n x} dx$ where $\omega_n = \frac{n\pi}{L}$.

[4] That is, for each ω_n there is a corresponding c_n where $|c_n|^2$ is a measure of the power of the sinusoids associated with ω_n.

[5] In this case the behavior of f must be known everywhere instead of on the interval $(-L, L)$.

[6] Kreyszig pg. 511

[7] Thus, if an input function has an even or odd symmetry then the transformed function shares the same symmetry.

[8] Thus, if an input function has symmetry then the Fourier transform is real-valued.

2. Calculate the following Fourier sine/cosine transformations. Please include the domain which the transformation is valid.

   (a) $\mathfrak{F}_c(e^{-ax}), \; a \in \mathbb{R}^+$
   (b) $\mathfrak{F}_c^{-1}\left(\frac{1}{1+\omega^2}\right)$
   (c) $\mathfrak{F}_s(e^{-ax}), \; a \in \mathbb{R}^+$
   (d) $\mathfrak{F}_s^{-1}\left(\sqrt{\frac{2}{\pi}}\frac{\omega}{a^2+\omega^2}\right), \; a \in \mathbb{R}^+$

3. Calculate the following transforms:

   (a) $\mathfrak{F}\{f\}$ where $f(x) = \delta(x - x_0), \; x_0 \in \mathbb{R}$.[9]
   (b) $\mathfrak{F}\{f\}$ where $f(x) = e^{-k_0|x|}, \; k_0 \in \mathbb{R}^+$.
   (c) $\mathfrak{F}^{-1}\left\{\hat{f}\right\}$ where $\hat{f}(\omega) = \frac{1}{2}\left(\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right), \; \omega_0 \in \mathbb{R}$.
   (d) $\mathfrak{F}^{-1}\left\{\hat{f}\right\}$ where $\hat{f}(\omega) = \frac{1}{2}\left(\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right), \; \omega_0 \in \mathbb{R}$.
   (e) Find $\hat{f}(\omega)$ where $f(x+c)$, $c \in \mathbb{R}$.

4. The convolution h of two functions f and g is defined as[10],

$$h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(p)g(x-p)dp = \int_{-\infty}^{\infty} f(x-p)g(p)dp. \tag{5}$$

   (a) Show that $\mathfrak{F}\{f * g\} = \sqrt{2\pi}\mathfrak{F}\{f\}\mathfrak{F}\{g\}$.
   (b) Find the convolution $h(x) = (f * g)(x)$ where $f(x) = \delta(x - x_0)$ and $g(x) = e^{-x}$.

5. Given the ODE,

$$y' + y = f(x), \quad 0 < x < \infty. \tag{6}$$

   (a) Calculate the frequency response associated with (6). [11]
   (b) Calculate the Green's function associated with (6).
   (c) Using convolution find the steady-state solution to the (6) for when $f(x) = \delta(x)$.

[9] Here the δ is the so-called Dirac, or continuous, delta function. This isn't a function in the true sense of the term but instead what is called a generalized function. The details are unimportant and in this case we care only that this Dirac-delta *function* has the property $\int_{-\infty}^{\infty} \delta(x-x_0)f(x)dx = f(x_0)$. For more information on this matter consider `http://en.wikipedia.org/wiki/Dirac_delta_function`. To drive home that this *function* is strange, let me spoil the punch-line. The sampling function $f(x) = \mathrm{sinc}(ax)$ can be used as a definition for the Delta *function* as $a \to 0$. So can a bell-curve probability distribution. Yikes!

[10] Here wee keep the same notation as Kreysig pg. 523

[11] this is often called the steady-state transfer function