Lecture 5. Electromagnetic waves

So far every E&B we've found that satisfies Maxwell's equations has involved some kind of non-field source (a σ or J)

But we don't have to have those. The Maxwell equations in vacuum (no σ or J) read:

$$\vec{\nabla}\cdot\vec{E}=0 \qquad \nabla\times\vec{E} = -\partial\vec{B}/\partial t$$
$$\vec{\nabla}\cdot\vec{B}=0 \qquad \nabla\times\vec{B} = \mu_0\varepsilon_0\,\partial\vec{E}/\partial t$$

We now have near-perfect symmetry between $\vec{E}$ + $\vec{B}$. Let's see if we can decouple these equations, that is, write them entirely in terms of only E or only B.

Consider $\nabla\times\vec{E} = -\partial\vec{B}/\partial t$. We can turn a $\vec{B}$ into an $\vec{E}$ via taking a curl, so curl both sides:

$$\nabla\times(\nabla\times\vec{E}) = \nabla\times(-\partial\vec{B}/\partial t)$$

Use a vector identity and the fact that time + space derivatives commute:

$$\nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E} = -\frac{\partial}{\partial t}(\nabla\times\vec{B})$$

And $\nabla\cdot\vec{E}=0$ in vacuum

$$\boxed{\nabla^2\vec{E} = \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}}$$

You might recognize this as structurally equivalent to the wave equation:

$$\nabla^2\varphi = \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}$$ For a wave with speed c.

Thus any E-field satisfying the above looks like a wave with speed $\frac{1}{\sqrt{\varepsilon_0\mu_0}} = c$.

And since people knew the values of ε_0, μ_0, and c for light, just like that Maxwell showed that light is an electromagnetic wave.

Why electromagnetic? Well, we can do the same thing starting from the Ampere-Maxwell eqn and get:

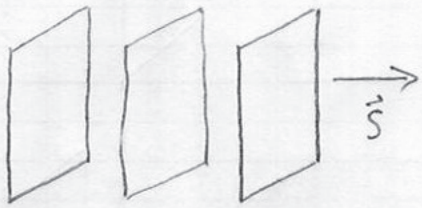
$$\boxed{\nabla^2\vec{B} = \mu_0\varepsilon_0\frac{\partial^2\vec{B}}{\partial t^2}}$$

Now, these equations for $\vec{E}$ + $\vec{B}$ are <u>linear</u>, meaning solutions will obey the superposition principle.

It's easy to check that one solution for $\vec{E}$ is $E_0 e^{i(kz-\omega t)}\hat{x}$ (we already know this from dealing with the wave equation in other contexts)

This is a very special solution because we know from Fourier analysis that functions of the form $e^{i(kz-\omega t)}$ can form a complete basis (like sines & cosines). So let's look at in more detail.

We call $E_0 e^{i(kz-\omega t)}\hat{\imath}$ a plane wave, because it has the same value everywhere in the x-y plane while it propagates in the z direction



Now, if we crank $E_0 e^{i(kz-\omega t)}\hat{\imath}$ through the wave equation, we get the condition

$$\omega = ck$$

This is the relationship between wavespeed, wavelength, and frequency.

This relation is called the dispersion relation, because later on this relationship will describe how different frequencies of light travel at different speeds through materials and thus spread out, or disperse.

Now, an equation describing a legitimate electromagnetic wave needs to also satisfy all the Maxwell equations individually. This is mostly just an exercise in crank turning, but let's look at a couple key points:

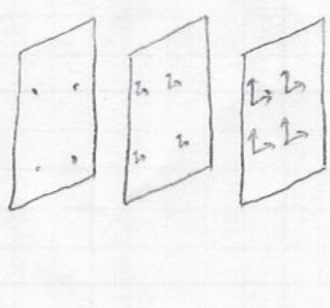
We can use $\nabla \times \vec{E} = -\partial \vec{B}/\partial t$ to recover the corresponding $\vec{B}$-field,

$\vec{B} = B_0 e^{i(kz-\omega t)}\hat{\jmath}$ with $B_0 = E_0/c$

You'll notice that E & B are mutually perpendicular, and are also perpendicular to the direction of propagation. This is consistent with $\vec{S} = \vec{E}\times\vec{B}/\mu_0$, which claims that any energy transfer from E & B fields will be orthogonal to both.

(Slide: classical representation of $\vec{E}$ & $\vec{B}$ for a plane wave)

vs.



We have just a few more things to say about plane waves. First, every wave we've written so far has been propagating in the z-direction. An E-field propagating in the y-direction would look like:

$$E_0 e^{i(ky-\omega t)}\,\hat{\imath}$$

And an $\vec{E}$-field propagating in some (mostly) arbitrary direction would look like:

$$E_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\,\hat{\imath}$$

With $\vec{k} = k_y\hat{\jmath} + k_z\hat{k}$ being the wavevector that describes the direction of propagation. $\vec{r}$ is the observation point, $x\hat{\imath}+y\hat{\jmath}+z\hat{k}$

Second, every $\vec{E}$-field we've written so far has been pointing in the $\hat{\imath}$ direction. This doesn't have to be the case. This is okay:

$$E_0 e^{i(kz-\omega t)}\,\hat{\jmath}$$

But this is not:

$$E_0 e^{i(kz-\omega t)}\,\hat{k}$$

Basically, as long as the field and propagation directions respect $\vec{S} = \vec{E}\times\vec{B}/\mu_0$, we're okay.

The direction that the fields point (not propagate) is referred to as polarization.

Linearly polarized light points along one well-defined direction, and is used in most modern displays. $E_0 e^{i(kz-\omega t)}\,\hat{\imath}$ is linearly polarized light.

"Natural" light is unpolarized: Basically a superposition of polarizations pointing in all directions.

More exotic options exist, such as circular polarization, which you'll work with in the homework.

A note on reality

We may represent waves as complex quantities, but in real life all observable quantities are strictly real. Ultimately we must take the real part of what we have.

Furthermore, while we can perform linear operations (linear in $\vec{E},\vec{B}$) on complex fields (such as $A\vec{E}_1 + B\vec{E}_2$), to perform nonlinear operations we must take the real part first (as with $\vec{E}\cdot\vec{E}$ or $\vec{E}\times\vec{B}$)