FOWLES "MODERN OPTICS

where k_x, k_y, and k_z are the components of **k**. Let A, B, and C be the linear dimensions of the cavity in the x, y, and z directions, respectively. Then a stationary pattern or *mode* will exist if the wave function is periodic in a manner expressed by the following equations:

$$k_xA = \pi n_x \qquad k_yB = \pi n_y \qquad k_zC = \pi n_z \tag{7.9}$$

where n_x, n_y, and n_z are integers. Each set (n_x, n_y, n_z) corresponds to a possible mode of the radiation in the cavity (Figure 7.3). Since $k^2 = k_x^2 + k_y^2 + k_z^2$, then

$$k^2 = \frac{\omega^2}{c^2} = \pi^2\left(\frac{n_x^2}{A^2} + \frac{n_y^2}{B^2} + \frac{n_z^2}{C^2}\right) \tag{7.10}$$

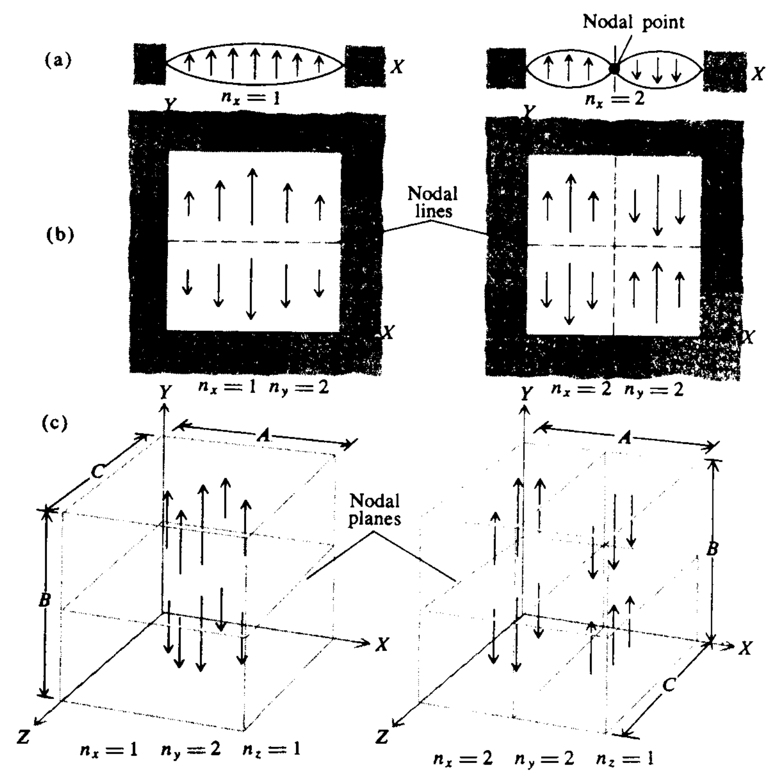

Figure 7.3. Standing wave patterns (modes) in various cavities. Part (a) shows the two lowest modes ($n = 1$ and $n = 2$) of a one-dimensional cavity. In (b) are shown the (1,2) and (2,2) modes of a two-dimensional cavity. Finally, in (c) the (1,2,1) and (2,2,1) modes of a three-dimensional cavity are illustrated.

or, equivalently,

$$\frac{4\nu^2}{c^2} = \frac{n_x^2}{A^2} + \frac{n_y^2}{B^2} + \frac{n_z^2}{C^2} \tag{7.11}$$

The above result shows that for a given frequency ν, only certain values of n_x, n_y, and n_z are allowed.

Let us examine Figure 7.4 in which Equation (7.11) is plotted

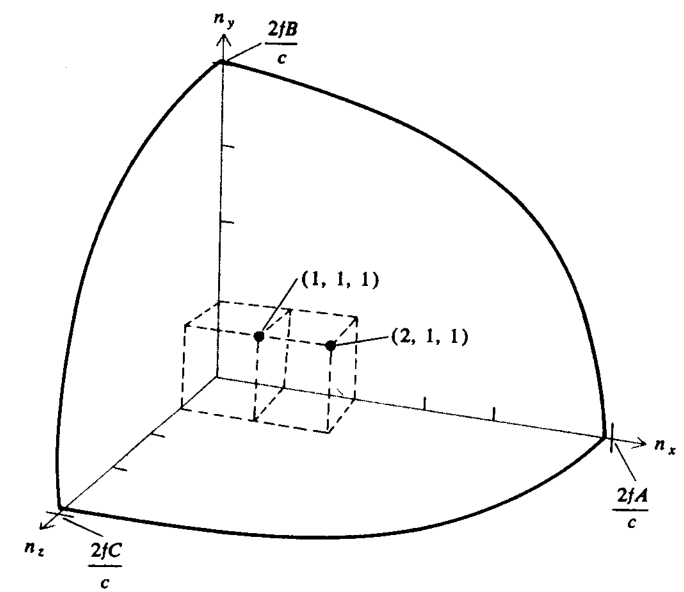

Figure 7.4. Unit cubes and associated points representing modes of a cavity. One octant of the limiting ellipsoid is shown.

graphically in terms of coordinates n_x, n_y, and n_z. The various modes are represented here by points at the corners of unit cubes, some of which are indicated in the figure. Now Equation (7.11) is the equation of an ellipsoid whose semiaxes are given by $2\nu A/c$, $2\nu B/c$, and $2\nu C/c$. The volume of one octant of this ellipsoid is therefore

$$\frac{1}{8}\,\frac{4\pi}{3}\,\frac{2\nu A}{c}\,\frac{2\nu B}{c}\,\frac{2\nu C}{c} = \frac{4\pi\nu^3 ABC}{3c^3} = \frac{4\pi\nu^3}{3c^3}\,V \tag{7.12}$$

where $V = ABC$ is the volume of the cavity. Since each unit cube is associated with one mode, the above expression is equal to the number of modes for *all frequencies equal to or less* than ν. Only one octant of the ellipsoid is needed to count the modes, because both positive and negative values of the ns correspond to the same mode.