Voltage, energy, and delta functions

At this point we have: $\nabla \cdot \vec{E} = \rho/\varepsilon_0$ and $\nabla \times \vec{E} = 0$

plus two boundary conditions that follow from those:

$$E_{1,\perp} - E_{2,\perp} = \sigma/\varepsilon_0$$

$$\vec{E}_{1,\parallel} - \vec{E}_{2,\parallel} = 0$$

And we can also derive Coulomb's law from Gauss's law:

$$\vec{E}(\vec{x}) = \int \frac{K\rho(\vec{x}')\, d^3x' \,(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

Strictly speaking, this is more than enough to do electrostatics. Knowing charge gives us the fields, which leads to forces via $\vec{F} = q\vec{E}$.

But as you may recall from intro physics, sometimes we prefer to cast things in terms of voltage and energy instead of field and force.

We know from math that given some curl-free field ($\nabla \times \vec{E}$ everywhere), we can define some scalar function V such that

$$\vec{E}(\vec{x}) = -\nabla V(\vec{x})$$

which goes by a few different names, including the voltage, the electric potential, and just the potential.

V has the same information as $\vec{E}$, but is a bit more pleasant to deal with for being a scalar, and also hooks into energy pretty directly.

We've seen before that: $\Delta U = q\Delta V$

And we've seen that we can construct V according to either:

$$\Delta V = -\int \vec{E} \cdot d\vec{\ell}$$ (a statement about differences in voltage between two points)

or $$V(\vec{x}) = \int \frac{K\rho(\vec{x}')\, d^3x'}{|\vec{x}-\vec{x}'|}$$ (a generalization of the potential from a point charge, $V = \frac{Kq}{r}$)

We can arrange the relationships between all these quantities in a handy little square:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\vec{F} \arrow[r, leftrightarrow, "{\vec{F} = -\nabla U}", "{\Delta U = -\int \vec{F}\cdot d\vec{l}}"'] \arrow[d, leftrightarrow, "{\vec{F} = q\vec{E}}"'] & U \arrow[d, leftrightarrow, "{\Delta U = q\Delta V}"] \\
\vec{E} \arrow[r, leftrightarrow, "{\vec{E} = -\nabla V}"', "{\Delta V = -\int \vec{E}\cdot d\vec{l}}"] & V
\end{tikzcd}$$

All of the above is from intro physics, so let's start adding some new stuff. We have both

$$\nabla \cdot \vec{E} = \rho/\varepsilon_0 \qquad \text{and} \qquad \vec{E} = -\nabla V$$

so substituting the latter into the former yields:

$$\nabla \cdot (-\nabla V) = \rho/\varepsilon_0 \quad \Rightarrow \quad \boxed{-\nabla^2 V = \rho/\varepsilon_0}$$

Which is Poisson's equation - the partial differential equation that yields V in electrostatics, given a known source ρ.

We'll be solving Poisson's equation a lot in the near future, and to solve PDEs we need boundary conditions. What might the BCs on V be?

Well, we know $\Delta V = -\int \vec{E}\cdot d\vec{l}$, and real E-fields are always finite, so for sufficently small $d\vec{l}$, $\Delta V \to 0$.

Thus $\boxed{V \text{ is always continuous}}$

That said, we know there can be discontinuities in $\overline{E}$ $\left(E_{1,\perp} - E_{2,\perp} = \sigma/\varepsilon_0\right)$, and E comes from derivatives of V, so if we're at some boundary and we let n indicate the direction normal to that boundary, we can write

$$E_{1,\perp} - E_{2,\perp} = \sigma/\varepsilon_0 \qquad \text{with} \quad E_\perp = -\frac{\partial V}{\partial n}$$

$$\Rightarrow \quad \boxed{\frac{\partial V_2}{\partial n} - \frac{\partial V_1}{\partial n} = \sigma/\varepsilon_0}$$

So while V is always continuous, derivatives of V aren't necessarily. At least, not the derivative perpendicular to a boundary. Since $E_\parallel$ is continuous, so must be $\frac{\partial V}{\partial \parallel}$, the derivative of V in a parallel direction.

(clicker question)

One more new-ish thing involving energy: We know the voltage made by a point source looks like:

$$V = \frac{Kq}{r}$$

And $\Delta U = q\Delta V$, so if we have one point charge of size q_1 sitting there at the origin and bring in another charge q_2 till its some distance r away, it'll cost us some energy:

$$U = \frac{K q_1 q_2}{r}$$

Thats the potential energy of an interaction between two point charges. If we have a collection of point charges, each pair counts, so we can write a total U as:

$$U_{tot} = \frac{1}{2} \sum_{i \neq j} \frac{K q_i q_j}{r_{ij}}$$

This is a double sum over both i and j, with a factor of ½ to compensate for counting each pair twice ($i=1, j=2$ is no different physically from $i=2, j=1$).

Generalizing over continuous objects, we break the object into points and integrate:

$$U_{tot} = \frac{1}{2}\int \frac{K\,dq_1\,dq_2}{r_{12}} = \boxed{\frac{1}{2}\int \frac{K\rho(\vec{x}_1)\rho(\vec{x}_2)\,d^3x_1\,d^3x_2}{|\vec{x}_1 - \vec{x}_2|}} \quad (1)$$

Note that the two ρ's represent the same charge distribution. We break the system into a bunch of dq's and check each against all the others.

As an alternative, note that $\int \frac{K\rho(\vec{x}_1)\,d^3x_1}{|\vec{x}_1 - \vec{x}_2|}$ is how

we'd write the voltage due to ρ at location $\vec{x}_2$ (the location of the second charge in any particular pair). Rewriting (1), we get:

$$\boxed{U_{tot} = \frac{1}{2}\int \rho(\vec{x})\,V(\vec{x})\,d^3x} \quad (2)$$

You can get the energy of the system by integrating the product of ρ and V everywhere.

Both of the above expressions explicitly reference charge. This is not shocking - we're used to potential energy being a thing associated with pairs of charges. But here's where it gets interesting:

Since $\nabla\cdot\vec{E} = \rho/\varepsilon_0$, $\rho = \varepsilon_0\,\nabla\cdot\vec{E}$. Sub that into (2):

$$U = \frac{1}{2}\varepsilon_0 \int \left(\nabla\cdot\vec{E}(\vec{x})\right) V(\vec{x})\,d^3x$$

Now, $\nabla\cdot(f\vec{A}) = f(\nabla\cdot\vec{A}) + \vec{A}\cdot(\nabla f)$

so $\nabla\cdot(V\vec{E}) = V(\nabla\cdot\vec{E}) + \vec{E}\cdot(\nabla V)$

$$\Rightarrow \quad U = -\frac{1}{2}\varepsilon_0 \int \vec{E}\cdot(\nabla V)\,d^3x + \frac{1}{2}\varepsilon_0 \int \nabla\cdot(V\vec{E})\,d^3x$$

Sub $\nabla V = -\vec{E}$ and apply the divergence theorem to the second term:

$$U = \frac{1}{2}\varepsilon_0 \int \vec{E}\cdot\vec{E}\, d^3x + \frac{1}{2}\varepsilon_0 \oint (V\vec{E})\cdot d\vec{A}$$

The integrals are over all space, so the second term examines V & $\vec{E}$ at the edge of all space. And for any real, finite source, that's zero, leaving:

$$\boxed{U = \frac{1}{2}\varepsilon_0 \int E^2\, d^3x} \qquad (3)$$

Which is entirely in terms of fields, not charge. So we can look at the fields themselves as being real things with real energy.

Just for fun, let's take a look at the energy associated with the field made by a point charge at the origin:

$$\vec{E} = \frac{Kq\,\hat{r}}{r^2} \qquad \text{and} \qquad U = \frac{1}{2}\int \vec{E}\cdot\vec{E}\, d^3x$$

$$\Rightarrow \quad U = \frac{1}{2}K^2q^2 \int \left(\frac{\hat{r}}{r^2}\cdot\frac{\hat{r}}{r^2}\right) d^3x$$

$$= \frac{K^2q^2}{2}\int \frac{1}{r^4}\, r^2 dr \sin\theta\, d\theta\, d\phi$$

$$= \frac{K^2q^2}{2}\cdot 4\pi \cdot \int_0^\infty \frac{1}{r^2}\, dr$$

$$= (\text{things}) \left[-\frac{1}{r}\Big|_0^\infty \right]$$ Which kind of diverges. That's bad.

Since $\vec{E}$-fields from point charges should contain infinite energy, it seems that point charges shouldn't be possible. But every experiment ever done indicates that an electron is a zero-radius true point. Fixing this apparent contradiction is one of the great achievements of quantum electrodynamics.

Day 3 - Delta functions, voltage, and energy

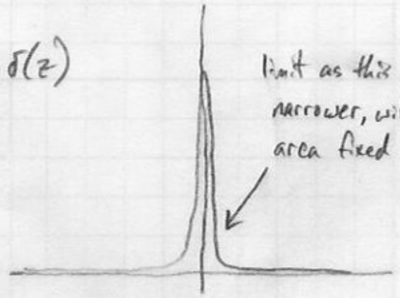
We deal with a lot of point sources in E+M, so let's review δ-functions

Fundamental definition: $\int_{-\infty}^{\infty} \delta(x) f(x) = f(0)$

Acts like a spike at $x=0$, 0 elsewhere

Works in higher dimensions, so $\delta^3(\vec{x}) = \delta(x)\delta(y)\delta(z)$

$\delta(x)$ is a function that picks out the value of $f(x)$ at $x=0$



So consider a point charge Q at the origin. In general,

$$\int \rho(\vec{x})\, d^3x' = Q$$

So what could $\rho(\vec{x})$ be? How about $Q\delta(\vec{x})$?

$$\int Q\delta(\vec{x})\, d^3x' = Q\int \delta(\vec{x}) \cdot 1 \cdot d^3x' = Q$$

Another useful result: $\frac{\hat{r}}{r^2}$ shows up a lot, and δ-functions help us work with this term.

What is $\vec{\nabla} \cdot \frac{\hat{r}}{r^2}$? We'll try direct calculation $\vec{\nabla} \cdot f(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 f_r\right)$

So $\vec{\nabla} \cdot \frac{\hat{r}}{r^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot \frac{1}{r^2}\right) = 0$

Seems fishy that div of ⇒ could be zero, but let's play along

That means $\int \nabla \cdot \frac{\hat{r}}{r^2}\, d^3x$ over some spherical volume including the origin is zero.

But div. thm says $\int \nabla \cdot \frac{\hat{r}}{r^2}\, d^3x = \oint \frac{\hat{r}}{r^2} \cdot d\vec{A} = \oint \frac{\hat{r}}{r^2} \cdot dA\hat{r}$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{1}{\cancel{r^2}} \cdot \cancel{r^2} \sin\theta \, d\theta \, d\phi$$

$$= 4\pi$$

Something's broken. Area version doesn't touch singularities, so we'll trust it.

Therefore $\int \nabla \cdot \frac{\hat{r}}{r^2}\, d^3x = 4\pi$ Given the singularity at the origin, this suggests

$\left(\vec{\nabla} \cdot \frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\vec{x})$, yielding $\int 4\pi\delta^3(\vec{x})\, d^3x = 4\pi$

First, best thing to do with this: Show Gauss's Law follows from Coulomb's Law

$$\vec{E} = \int K\rho(x') \frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} d^3x' \qquad \left(\int \frac{K\rho dV}{r^3}\vec{r} \right)$$

$$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\int \frac{K\rho(x')(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} d^3x'$$

$\vec{\nabla}$ is w.r.t. x, not dummy variable x'

$$= \int K\rho(x') \left[\vec{\nabla}\cdot \text{stuff}\right] d^3x'$$

$$= \int K\rho(x') 4\pi\delta^3(\vec{x}-\vec{x}') d^3x' = 4\pi K \rho(\vec{x})$$

And $4\pi k = \frac{1}{\varepsilon_0}$

$$\boxed{\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0}$$

Voltage, or electric potential, or just potential.

Know from math that if $\vec{\nabla}\times\vec{f} = 0$, can define unique (up to a constant) scalar function $g \ni -\vec{\nabla}g = f$

And scalars are nice, so we define electric potential via

$$\boxed{-\vec{\nabla}V = \vec{E}}$$

Scalar fcn that has all the info E does.
Similar to, given irrotational force field, $\vec{F} = -\nabla U$ from mech

Book goes one further, takes div of each side to get

$$-\vec{\nabla}\cdot\vec{\nabla}V = \vec{\nabla}\cdot\vec{E} = \boxed{-\nabla^2 V = \rho/\varepsilon_0 \quad \text{Poisson's eqn}}$$

Fundamental diff Eq defining V. Book makes big deal, then does nothing with it we couldn't do in Phys 200. But we'll come back to it.

For now, recall that $V_{point} = \frac{Kq}{r}$, so $\boxed{V_{general} = \int \frac{Kdq}{r}}$

Alternately, construct V via $\boxed{\Delta V = -\int \vec{E}\cdot d\vec{l}}$

Irrotational $\vec{E} \Rightarrow$ path independence

Like $\vec{E}$, $\boxed{\text{V has boundary conditions}}$

If $\Delta V = -\int \vec{E}\cdot d\vec{l}$, then as we integrate over a shorter and shorter path, $\Delta V \to 0$, so

$\boxed{\text{V is continuous always}}$