Problem 54:

The Boundary Value Problems now become

$$(\mathrm{I})\quad \begin{cases} X'' + kX = 0 \\ X'(0) = 0,\ X'(L_x) = 0 \end{cases}$$

and

$$(\mathrm{II})\quad \begin{cases} Y'' + pY = 0 \\ Y'(0) = 0,\ Y'(L_y) = 0 \end{cases}$$

which give rise to the Fourier modes,

$$X_n(x) = \cos(\sqrt{k_n}\,x),\quad \sqrt{k_n} = \frac{n\pi}{L_x},\ n = 0, 1, 2, \cdots$$

$$Y_m(y) = \cos(\sqrt{p_m}\,y),\quad \sqrt{p_m} = \frac{m\pi}{L_y},\ m = 0, 1, 2, 3, \ldots$$

and general sol$\underline{n}$

$$u(x,y,t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} B_{nm}\, e^{-\lambda_{nm} c^2 t} \cos\left(\frac{n\pi}{L_x}x\right)\cos\left(\frac{m\pi}{L_y}y\right)$$

Thus,

$$u(x,y,0) = f(x,y) = \sum_{m=0}^{\infty} \underbrace{\sum_{n=0}^{\infty} B_{nm} \cancelto{1}{e^{-\lambda_{nm} c^2 (0)}} \cos\left(\frac{n\pi}{L_x} x\right)}_{A_m} \cdot \cos\left(\frac{m\pi}{L_y} y\right) =$$

$$= \sum_{m=0}^{\infty} A_m \cos\left(\frac{m\pi}{L_y} y\right)$$

$$\Rightarrow \quad A_m = \frac{2}{L_y} \int_0^{L_y} f(x,y) \cos\left(\frac{m\pi}{L_y} y\right) dy, \quad m \neq 0$$

$$A_0 = \frac{1}{L_y} \int_0^{L_y} f(x,y)\, dy$$

<u>Note:</u>

$$A_m = \sum_{n=0}^{\infty} B_{nm} \cos\left(\frac{n\pi}{L_x} x\right)$$

$$\Rightarrow \quad B_{nm} = \frac{2}{L_x} \int_0^{L_x} A_m \cos\left(\frac{n\pi}{L_x} x\right) dx, \quad n \neq 0$$

$$\Rightarrow B_{0m} = \frac{1}{L_x}\int_0^{L_x} A_{m0}\,dx$$

$\Rightarrow$

$$B_{00} = \frac{1}{L_x}\int_0^{L_x} \frac{1}{L_y}\int_0^{L_y} f(x,y)\,dy\,dx$$

$$B_{n0} = \frac{2}{L_x L_y}\int_0^{L_x}\int_0^{L_y} f(x,y)\cos\left(\frac{n\pi}{L_x}x\right)dy\,dx$$

$$B_{0m} = \frac{2}{L_x L_y}\int_0^{L_x}\int_0^{L_y} f(x,y)\cos\left(\frac{m\pi}{L_y}y\right)dy\,dx$$

$$B_{nm} = \frac{4}{L_x L_y}\int_0^{L_x}\int_0^{L_y} f(x,y)\cos\left(\frac{m\pi}{L_y}y\right)\cos\left(\frac{n\pi}{L_x}x\right)dy\,dx$$

Also:

$$\lambda_{nm} = \left(\frac{n\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_y}\right)^2, \quad n = 0, 1, 2, \ldots$$
$$m = 0, 1, 2, \ldots$$

$$\Rightarrow \lambda_{00} = 0 \Rightarrow G_{00}(t) = \cancel{B_{00}}\, e^{-0t} = 1$$

is the only time dynamic that does not decay.

$$\Rightarrow u(x,y,t) \longrightarrow B_{00} \text{ as } t \to \infty$$