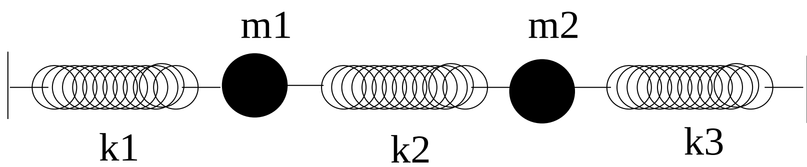


Figure 1.10: Two masses coupled by a spring and attached to walls.

This splitting of the degenerate frequency by an external magnetic field is called the Zeeman effect, after its discoverer Pieter Zeeman was born in May 1865, at Zonnemaire, a small village in the isle of Schouwen, Zeeland, The Netherlands. Zeeman was a student of the great physicists Onnes and Lorentz in Leyden. He was awarded the Nobel Prize in Physics in 1902. Zeeman succeeded Van der Waals (another Nobel prize winner) as professor and director of the Physics Laboratory in Amsterdam in 1908. In 1923 a new laboratory was built for Zeeman that included a quarter-million kilogram block of concrete for vibration free measurements.

We could continue the analysis by plugging these frequencies back into the amplitude equations 1.1.35. As an exercise, do this and show that the motion of the electron (and hence the electric field) is circularly polarized in the direction perpendicular to the magnetic field.

1.2 Two Coupled Masses

With only one mass and one spring, the range of motion is somewhat limited. There is only one characteristic frequency $\omega_0^2 = \frac{k}{m}$ so in the absence of damping, the transient (unforced) motions are all of the form $\cos(\omega_0 t + \Delta)$.

Now let us consider a slightly more general kind of oscillatory motion. Figure 1.10 shows two masses (m_1 and m_2) connected to fixed walls with springs k_1 and k_3 and connected to one another by a spring k_2. To derive the equations of motion, let's focus attention on one mass at a time. We know that for any given mass, say m_i (whose displacement from equilibrium we label x_i) it must be that

$$m_i \ddot{x}_i = F_i \tag{1.2.1}$$

where F_i is the total force acting on the ith mass. No matter how many springs and masses we have in the system, the force applied to a given mass must be transmitted by the two springs it is connected to. And the force each of these springs transmits is governed by the extent to which the spring is compressed or extended.

Referring to Figure 1.10, spring 1 can only be compressed or extended if mass 1 is displaced from its equilibrium. Therefore the force applied to m_1 from k_1 must be $-k_1 x_1$, just as before. Now, spring 2 is compressed or stretched depending on whether $x_1 - x_2$ is positive or not. For instance, suppose both masses are displaced to the right (positive x_i) with mass 1 being displaced more than mass 2. Then spring 2 is compressed relative to its equilibrium length and the force on mass 1 will in the negative x direction so as to restore the mass to its equilibrium position. Similarly, suppose both masses are displaced to the right, but now with mass 2 displaced more than mass 1, corresponding to spring 2 being stretched. This should result in a force on mass 1 in the positive x direction since the mass is being pulled away from its equilibrium position. So the proper expression of Hooke's law in any case is

$$m_1 \ddot{x}_1 = -k_1 x_1 - k_2(x_1 - x_2). \tag{1.2.2}$$

And similarly for mass 2

$$m_2 \ddot{x}_2 = -k_3 x_2 - k_2(x_2 - x_1). \tag{1.2.3}$$

These are the general equations of motion for a two mass/three spring system. Let us simplify the calculations by assuming that both masses and all three springs are the same. Then we have

$$\begin{aligned} \ddot{x}_1 &= -\frac{k}{m} x_1 - \frac{k}{m}(x_1 - x_2) \\ &= -\omega_0^2 x_1 - \omega_0^2 (x_1 - x_2) \\ &= -2\omega_0^2 x_1 + \omega_0^2 x_2. \end{aligned} \tag{1.2.4}$$

and

$$\begin{aligned} \ddot{x}_2 &= -\frac{k}{m} x_2 - \frac{k}{m}(x_2 - x_1) \\ &= -\omega_0^2 x_2 - \omega_0^2 (x_2 - x_1) \\ &= -2\omega_0^2 x_2 + \omega_0^2 x_1. \end{aligned} \tag{1.2.5}$$

Assuming trial solutions of the form

$$x_1 = Ae^{i\omega t} \tag{1.2.6}$$

$$x_2 = Be^{i\omega t} \tag{1.2.7}$$

we see that

$$(-\omega^2 + 2\omega_0^2)A = \omega_0^2 B \tag{1.2.8}$$

$$(-\omega^2 + 2\omega_0^2)B = \omega_0^2 A. \tag{1.2.9}$$

Substituting one into the other we get

$$A = \frac{\omega_0^2}{2\omega_0^2 - \omega^2} B, \tag{1.2.10}$$

and therefore

$$(2\omega_0^2 - \omega^2)B = \frac{\omega_0^4}{2\omega_0^2 - \omega^2}B. \tag{1.2.11}$$

This gives an equation for ω^2

$$(2\omega_0^2 - \omega^2)^2 = \omega_0^4. \tag{1.2.12}$$

There are two solutions of this equation, corresponding to $\pm\omega_0^2$ when we take the square root. If we choose the plus sign, then

$$2\omega_0^2 - \omega^2 = \omega_0^2 \Rightarrow \omega^2 = \omega_0^2. \tag{1.2.13}$$

On the other hand, if we choose the minus sign, then

$$2\omega_0^2 - \omega^2 = -\omega_0^2 \Rightarrow \omega^2 = 3\omega_0^2. \tag{1.2.14}$$

We have discovered an important fact: spring systems with two masses have two characteristic frequencies. We will refer to the frequency $\omega^2 = 3\omega_0^2$ as "fast" and $\omega^2 = \omega_0^2$ as "slow". Of course these are relative terms. Now that we have the frequencies we can investigate the amplitude. First, since

$$A = \frac{\omega_0^2}{2\omega_0^2 - \omega^2}B, \tag{1.2.15}$$

we have for the slow mode ($\omega = \omega_0$)

$$A = B, \tag{1.2.16}$$

which corresponds to the two masses moving in phase with the same amplitude. On the other hand, for the fast mode

$$A = -B. \tag{1.2.17}$$

For this mode, the amplitudes of the two mass' oscillation are the same, but they are out of phase. These two motions are easy to picture. The slow mode corresponds to both masses moving together, back and forth, as in Figure 1.11 (bottom). The fast mode corresponds to the two masses oscillating out of phase as in Figure 1.11 (top).

1.2.1 A Matrix Appears

There is a nice way to simplify the notation of the previous section and to introduce a powerful mathematical at the same time. Let's put the two displacements together into a vector. Define a vector $\mathbf{u}$ with two components, the displacements of the first and second mass:

$$\mathbf{u} = \begin{bmatrix} Ae^{i\omega t} \\ Be^{i\omega t} \end{bmatrix} = e^{i\omega t}\begin{bmatrix} A \\ B \end{bmatrix}. \tag{1.2.18}$$

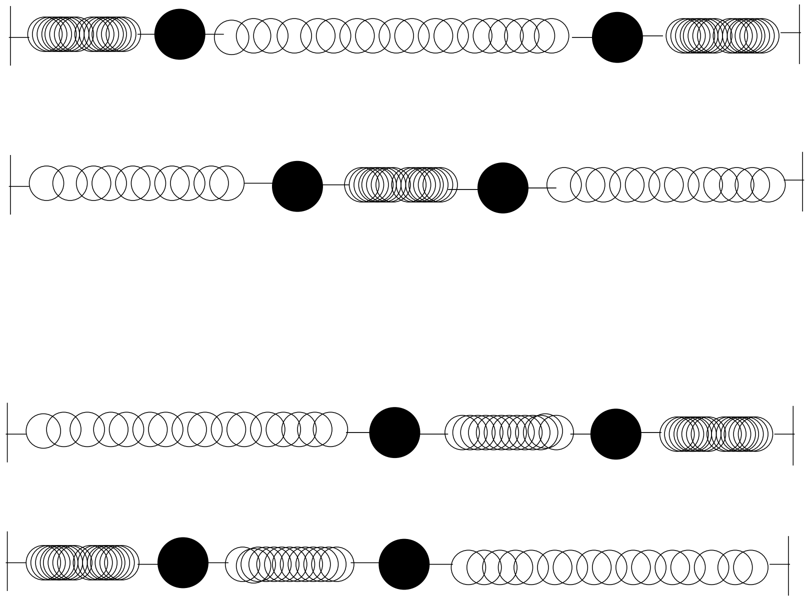


Figure 1.11: With two coupled masses there are two characteristic frequencies, one "slow" (bottom) and one "fast" (top).

We've already seen that we can multiply any solution by a constant and still get a solution, so we might as well take A and B to be equal to 1. So for the slow mode we have

$$\mathbf{u} = e^{i\omega_0 t} \left[\begin{array}{c} 1 \\ 1 \end{array} \right], \tag{1.2.19}$$

while for the fast mode we have

$$\mathbf{u} = e^{i\sqrt{3}\omega_0 t} \left[\begin{array}{c} 1 \\ -1 \end{array} \right]. \tag{1.2.20}$$

Notice that the amplitude part of the two modes

$$\left[\begin{array}{c} 1 \\ 1 \end{array} \right] \text{ and } \left[\begin{array}{c} 1 \\ -1 \end{array} \right] \tag{1.2.21}$$

are **orthogonal**. That means that the dot product of the two vectors is zero: $1 \times 1 + 1 \times (-1) = 0$.[6] As we will see in our discussion of linear algebra, this means that the two vectors point at right angles to one another. This orthogonality is an absolutely fundamental property of the natural modes of vibration of linear mechanical systems.

[6]
$$\left[\begin{array}{c} 1 \\ 1 \end{array} \right] \cdot \left[\begin{array}{c} 1 \\ -1 \end{array} \right] \equiv [1, 1] \left[\begin{array}{c} 1 \\ -1 \end{array} \right] = 1 \cdot 1 - 1 \cdot 1 = 0.$$

1.2.2 Matrices for two degrees of freedom

The equations of motion are (see Figure 1.10):

$$m_1\ddot{x}_1 + k_1 x_1 + k_2(x_1 - x_2) = 0 \tag{1.2.22}$$
$$m_2\ddot{x}_2 + k_3 x_2 + k_2(x_2 - x_1) = 0. \tag{1.2.23}$$

We can write these in matrix form as follows.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{1.2.24}$$

Or, defining a mass matrix

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \tag{1.2.25}$$

and a "stiffness" matrix

$$K = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix} \tag{1.2.26}$$

we can write the matrix equation as

$$M\ddot{\mathbf{u}} + K\mathbf{u} = \mathbf{0} \tag{1.2.27}$$

where

$$\mathbf{u} \equiv \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{1.2.28}$$

This is much cleaner than writing out all the components and has the additional advantage that we can add more masses/springs without changing the equations, we just have to incorporate the additional terms into the definition of M and K.

Notice that the mass matrix is always invertible since it's diagonal and all the masses are presumably nonzero. Therefore

$$M^{-1} = \begin{bmatrix} {m_1}^{-1} & 0 \\ 0 & {m_2}^{-1} \end{bmatrix}. \tag{1.2.29}$$

So we can also write the equations of motion as

$$\ddot{\mathbf{u}} + M^{-1}K\mathbf{u} = \mathbf{0}. \tag{1.2.30}$$

And it is easy to see that

$$M^{-1}K = \begin{bmatrix} \frac{k_1+k_2}{m_1} & \frac{-k_2}{m_1} \\ \frac{-k_2}{m_2} & \frac{k_2+k_3}{m_2} \end{bmatrix}.$$

As another example, let's suppose that all the masses are the same and that $k_1 = k_3 = k$. Letting $\omega_0 = \sqrt{k/m}$ as usual and defining $\Omega = \sqrt{k_2/m}$, we have the following beautiful form for the matrix $M^{-1}K$:

$$M^{-1}K = \Omega^2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \omega_0^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{1.2.31}$$

In the limit that Ω goes to zero the coupling between the masses becomes progressively weaker. If $\Omega = 0$, then the equations of motion reduce to those for two uncoupled oscillators with the same characteristic frequency ω_0.

1.2.3 The energy method

In this example of two coupled masses, it's not entirely trivial to keep track of how the two masses interact. Unfortunately, we're forced into this by the Newtonian strategy of specifying forces explicitly. Fortunately this is not the only way to skin the cat. For systems in which energy conserved (no dissipation, also known as **conservative** systems), the force is the gradient of a potential energy function.[7]

Since energy is a scalar quantity it is almost always a lot easier to deal with than the force itself. In our 1-D system of masses and springs, that might not be apparent, but even so using energy simplifies life significantly. Think about it: the potential energy of the system must be the sum of the potential energies of the individual springs. And the potential energy of a spring is the spring constant times the square of amount the spring is compresses or extended. So the potential energy of the system is just $\frac{1}{2}\left[k_1 x_1^2 + k_2(x_2 - x_1)^2 + k_3 x_2^2\right]$. Unlike when dealing with the forces, it doesn't matter whether we write the second term as $x_2 - x_1$ or $x_1 - x_2$ since it gets squared.

The energy approach is easily extended to an arbitrary number of springs and masses. It's up to us to define just what the system will be. For instance do we connect the end springs to the wall, or do we connect the end masses? It doesn't matter much except in the labels we use and the limits of the summation. For now we will assume that we have n springs, the end springs being connected to rigid walls, and $n-1$ masses. So, $n-1$ masses $\{m_i\}_{i=1,n-1}$ and n spring constants $\{k_i\}_{i=1,n}$. Then the total energy is

$$E = \text{K.E.} + \text{P.E.} = \frac{1}{2}\sum_{i=1}^{n-1} m_i \dot{x}_i^2 + \frac{1}{2}\sum_{i=1}^{n} k_i (x_i - x_{i-1})^2. \tag{1.2.32}$$

[7] The work done by a force in displacing a system from a to b is $\int_a^b F\, dx$. If $F = -\frac{dU}{dx}$, then $\int_a^b F\, dx = -\int dU = -[U(b) - U(a)]$. In other words the work depends only on the endpoints, not the path taken. In particular, if the starting and ending point is the same, the work done is zero. This is true in 3 dimensions too where it is easier to visualize complicated paths.

To derive the equations of motion, all we have to do is set $m_j \ddot{x}_j = -\frac{\partial U}{\partial x_j}$. Taking the derivative is slightly tricky. Since j is arbitrary (we want to be able to study any mass), there will be two nonzero terms in the derivative of U, corresponding to the two situations in which one of the terms in the sum is equal to x_j. This will happen when

- $i = j$, in which case the derivative is $k_j(x_j - x_{j-1})$.
- $i - 1 = j$, in which case $i = j + 1$ and the derivative is $-k_{j+1}(x_{j+1} - x_j)$.

Putting these two together we get

$$m_j \ddot{x}_j = -\frac{\partial U}{\partial x_j} = k_{j+1}(x_{j+1} - x_j) - k_j(x_j - x_{j-1}). \tag{1.2.33}$$

Once you get the hang of it, you'll see that in most cases the energy approach is a lot easier than dealing directly with the forces. After all, force is a vector, while energy is always a scalar. For now, let's simplify Equation 1.2.33 by taking all the masses to be the same m and all the spring constants to be the same k. Then, using $\omega_0^2 = k/m$ again, we have

$$\frac{1}{\omega_0^2} \ddot{x}_j = x_{j+1} - 2x_j + x_{j-1}. \tag{1.2.34}$$

1.2.4 Matrix form of the coupled spring/mass system

We can greatly simplify the notation of the coupled system using matrices. Let's consider the n mass case in Equation 1.2.34. We would like to be able to write this as

$$\frac{1}{\omega_0^2} \ddot{\mathbf{u}} \equiv \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ . \\ . \\ . \\ \ddot{x}_{n-1} \end{bmatrix} = \text{ some matrix dotted into } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ . \\ x_{n-1} \end{bmatrix} \equiv \mathbf{u}. \tag{1.2.35}$$

The symbol $\equiv$ means the two things on either side are equal by definition.

Looking at Equation 1.2.34 we can see that this matrix must couple each mass to its nearest neighbors, with the middle mass getting a weight of -2 and the neighboring masses getting weights of 1. Thus the matrix must be

$$\begin{bmatrix} -2 & 1 & 0 & 0 & \dots \\ 1 & -2 & 1 & 0 & \dots \\ 0 & 1 & -2 & 1 & \dots \\ \vdots & & & \ddots & \\ 0 & \dots & 0 & 1 & -2 \end{bmatrix}. \tag{1.2.36}$$

So we have

$$\frac{1}{\omega_0^2}\ddot{\mathbf{u}} = \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \cdot \\ \cdot \\ \cdot \\ \ddot{x}_{n-1} \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 & 0 & \dots \\ 1 & -2 & 1 & 0 & \dots \\ 0 & 1 & -2 & 1 & \dots \\ \vdots & & & \ddots & \\ 0 & \dots & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \end{bmatrix} = \mathbf{u}. \tag{1.2.37}$$

If we denote the matrix by K, then we collapse these n coupled second order differential equations to the following beautiful vector differential equation.

$$\frac{1}{\omega_0^2}\ddot{\mathbf{u}} = K \cdot \mathbf{u}. \tag{1.2.38}$$

We don't yet have the mathematical tools to analyze this equation properly, that is why we will spend a lot of time studying linear algebra. However we can proceed. Surprisingly enough if we add even more springs and masses to our system, we will get an equation we can solve analytically, but we need to an an infinite number of them! Let's see how we can do this.

First, let's be careful how we interpret the dependent and independent variables. If I write the vector of displacements from equilibrium as $\mathbf{u}$, then its components are $(\mathbf{u})_i \equiv x_i$. Let's forget about x and think only of displacements $\mathbf{u}$ or $(\mathbf{u})_i$. The reason is we want to be able to use x as a variable to denote the position along the spring/mass lattice at which we are measuring the displacement. Right now, with only a finite number of masses, we are using the index i for this purpose. But we want to let i go to infinity and have a continuous variable for this; this is what we will henceforth use x for. But before we do that, let's look at how we can approximate the derivative of a function. Suppose $f(x)$ is a differentiable function. Then, provided h is small

$$f'(x) \approx \frac{f(x+\frac{h}{2}) - f(x-\frac{h}{2})}{h}. \tag{1.2.39}$$

We can do this again for each of the two terms on the right hand side and achieve an approximation for the second derivative:

$$\begin{aligned} f''(x) &\approx \frac{f(x+h) - f(x)}{h^2} - \frac{f(x) - f(x-h)}{h^2} \\ &= \frac{1}{h^2}\left(f(x+h) - 2f(x) + f(x-h)\right). \end{aligned} \tag{1.2.40}$$

Now suppose that we want to look at this approximation to f'' at points x_i along the x-axis. For instance, suppose we want to know $f''(x_i)$ and suppose the distance between the x_i points is constant and equal to h. Then

$$f''(x_i) \approx \frac{1}{h^2}\left(f(x_{i+1}) - 2f(x_i) + f(x_{i-1})\right). \tag{1.2.41}$$

Or, if we denote $f(x_i)$ by f_i, then the approximate second derivative of the function at a given i location looks exactly like the ith row of the matrix above. In the limit that the number of mass points (and hence i locations) goes to infinity, the displacement $\mathbf{u}$ becomes a continuous function of the spatial location, which we now refer to as x, and K becomes a second derivative operator. To get the limit we have to introduce the lattice spacing h:

$$\frac{1}{\omega_0^2}\ddot{\mathbf{u}} = h^2 \frac{1}{h^2} K \cdot \mathbf{u}. \tag{1.2.42}$$

We can identify each row of $\frac{1}{h^2}K \cdot \mathbf{u}$ as being the approximate second derivative of the corresponding displacement. But we can't quite take the limit yet, since ω_0 is defined in terms of the discrete mass and it's not clear what this would mean in the limit of a continuum. So let's write this as

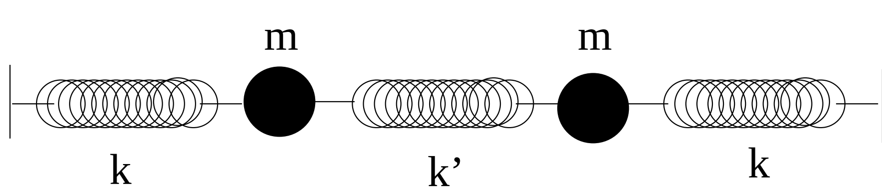
$$\ddot{\mathbf{u}} = \frac{k}{m}\frac{h^3}{h^3} K \cdot \mathbf{u} = \frac{k}{h}\frac{h^3}{m}\frac{1}{h^2} K \cdot \mathbf{u} \tag{1.2.43}$$

so that in the limit that the number of mass points goes to infinity, but the mass of each point goes to zero and the spacing h goes to zero, we can identify $\frac{m}{h^3}$ as the density and $\frac{k}{h}$ as the stiffness per unit length. Let's call the latter E. Now in this limit $\mathbf{u}$ is no longer a finite length vector, but a continuous function of the position x. Since it is also a function of time, these derivatives must become partial derivatives. So in this limit we end up with

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 u(x,t)}{\partial x^2}. \tag{1.2.44}$$

This is called the wave equation.

Exercises



1.1 Write down the equations of motion for the system above in terms of the displacements of the two masses from their equilibrium positions. Call these displacements x_1 and x_2.

Answer: The equations of motion are