PHGN 462-507, Quiz 1 **Key**

1. Let $f \equiv \mathrm{e}^{\mathrm{i}\omega_1 t} + \mathrm{e}^{\mathrm{i}\omega_2 t}$, where ω_1, ω_2, and t are real. Express $|f|^2$ as a real function.

   There are two ways to do this problem. The first method to solving this problem is to observe that $|f|^2 = f \cdot f^*$, where f^* is the complex conjugate of f. This conjugate can be written as

   $$f^* = \mathrm{e}^{-\mathrm{i}\omega_1 t} + \mathrm{e}^{-\mathrm{i}\omega_2 t},$$

   which, when multiplied by f, becomes

   $$|f|^2 = \left(\mathrm{e}^{\mathrm{i}\omega_1 t} + \mathrm{e}^{\mathrm{i}\omega_2 t}\right)\left(\mathrm{e}^{-\mathrm{i}\omega_1 t} + \mathrm{e}^{-\mathrm{i}\omega_2 t}\right),$$

   which becomes,

   $$|f|^2 = 2 + \underbrace{\mathrm{e}^{\mathrm{i}(\omega_1-\omega_2)t} + \mathrm{e}^{-\mathrm{i}(\omega_1-\omega_2)t}}_{2\cos((\omega_1-\omega_2)t)},$$

   and finally

   $$\boxed{|f|^2 = 2[1 + \cos((\omega_1 - \omega_2)t)].}$$

   The second method to solving this problem is to expand the exponential functions into cosine and sine function with Euler's formula,

   $$\mathrm{e}^{\mathrm{i}\varphi} = \cos\varphi + \mathrm{i}\sin\varphi,$$

   and then adding the real and imaginary parts of f in quadrature. Expanding f and separating the real and imaginary parts gives

   $$f = \underbrace{[\cos(\omega_1 t) + \cos(\omega_2 t)]}_{\text{Real}} + \mathrm{i}\underbrace{[\sin(\omega_1 t) + \sin(\omega_2 t)]}_{\text{Imaginary}}.$$

   Adding the real and imaginary parts in quadrature gives

   $$|f|^2 = [\cos(\omega_1 t) + \cos(\omega_2 t)]^2 + [\sin(\omega_1 t) + \sin(\omega_2 t)]^2,$$

   which expands to

   $$|f|^2 = \cos^2(\omega_1 t) + 2\cos(\omega_1 t)\cos(\omega_2 t) + \cos^2(\omega_2 t) + \sin^2(\omega_1 t) + 2\sin(\omega_1 t)\sin(\omega_2 t) + \sin^2(\omega_2 t)$$

   By observing that

   $$\sin^2\phi + \cos^2\phi = 1$$

   and

   $$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta,$$

   we, again, obtain

   $$\boxed{|f|^2 = 2[1 + \cos((\omega_1 - \omega_2)t)].}$$

2. Let $x \ll 1$. Expand the quantity

$$g(x) \equiv \frac{1}{1+\mathrm{e}^{-x}}$$

about $x = 0$, with x in the numerator.

A Taylor series approximation is a method used to express any nonlinear function, like g, in the form of an infinite polynomial. While an infinite sum may not look attractive at first glance, fortunately, one can truncate the series, leaving only a few terms. The infinite sum looks like so,

$$g(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{\mathrm{d}^n}{\mathrm{d}x^n} g(x)\right]_{x=x_0} (x-x_0)^n.$$

In our case, we will expand about $x_0 = 0$, which means that the linear series will be accurate for values close to $x = 0$. Our truncated expansion becomes

$$g(x) \approx g(0) + g'(0)x + O(x^2),$$

where the $O(x^2)$ notes that we have truncated all terms of order x^2 and higher. We trivially find that

$$g(0) = \left[\frac{1}{1+\mathrm{e}^{-x}}\right]_{x=0} = \frac{1}{2}.$$

We find that its derivative is

$$g'(0) = \left[\frac{\mathrm{e}^{-x}}{(1+\mathrm{e}^{-x})^2}\right]_{x=0} = \frac{1}{4},$$

which makes the linear approximation,

$$\boxed{g(x) \approx \frac{1}{2} + \frac{1}{4}x + O(x^2).}$$

There is another, easier method to solve this problem by expanding the denominator of g to obtain

$$g \approx \frac{1}{2-x},$$

which becomes

$$g \approx \frac{1}{2}\frac{1}{1-\frac{1}{2}x}.$$

The fraction can be expanded to, again, produce

$$g \approx \frac{1}{2}\left(1 + \frac{1}{2}x + O(x^2)\right),$$

which gives us

$$\boxed{g(x) \approx \frac{1}{2} + \frac{1}{4}x + O(x^2).}$$