

Homework #2 Solution:

1. We have the polynomial

$$p(t) = a_0 + a_1t + a_2t^2$$

and the data points  $(1, 12), (2, 15), (3, 16)^{(\star)}$ . This generates 3 linear equations

$$\begin{aligned} p(1) &= a_0 + a_1 + a_2 = 12 \\ p(2) &= a_0 + 2a_1 + 4a_2 = 15 \\ p(3) &= a_0 + 3a_1 + 9a_2 = 16 \end{aligned}$$

and the corresponding augmented matrix.

$$\begin{aligned} & \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 1 & 2 & 4 & 15 \\ 1 & 3 & 9 & 16 \end{array} \right] \xrightarrow[\sim]{\substack{R3=R3-R1 \\ R2=R2-R1}} \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 2 & 8 & 4 \end{array} \right] \sim \\ & \xrightarrow{\sim R3=R3-2R2} \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 2 & -2 \end{array} \right] \xrightarrow{\sim R3=R3/2} \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & -1 \end{array} \right] \sim \\ & \xrightarrow[\sim]{\substack{R1=R1-R3 \\ R2=R2-3R3}} \left[ \begin{array}{ccc|c} 1 & 1 & 0 & 13 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{array} \right] \xrightarrow{\sim R1=R1-R2} \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{array} \right] \sim \end{aligned}$$

The row equivalent linear system is then,

$$\begin{aligned} a_0 &= 7 \\ a_1 &= 6 \\ a_2 &= -1 \end{aligned}$$

which implies that  $p(t) = 7 + 6t - t^2$  is the quadratic polynomial which indicates  $(\star)$ .

2. If  $\vec{x} = [1 \ 0]^T$  then the new vectors  $A(\theta)\vec{x}$  will correspond to the vectors on the unit circle.  $A$  represents rigid (norm-preserving) rotations (counter clockwise) of  $\vec{x}$ .  $A = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$  represents clockwise rotations.

3.  $A = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}$  Determine  $A^{-1}$  via  
a) calculate  $\det(A)$ .

$$\begin{aligned} \det(A) &= 3\det \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} - 0 \cdot \det \begin{pmatrix} 6 & 7 \\ 3 & 4 \end{pmatrix} + 2\det \begin{pmatrix} 6 & 7 \\ 2 & 1 \end{pmatrix} \\ &= 3(5) - 0(3) + 2(-8) = 15 - 16 = -1 \end{aligned}$$

b) The Gauss-Jordan Method

$$\begin{aligned}
 & \left[ \begin{array}{ccc|ccc} 3 & 6 & 7 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} R1 - 3R2 \\ 2R1 - 3R3 \end{array} \sim \left[ \begin{array}{ccc|ccc} 3 & 0 & 4 & 1 & -3 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 3 & 2 & 2 & 0 & -3 \end{array} \right] \begin{array}{l} \\ 2R3 - 3R2 \end{array} \\
 & \sim \left[ \begin{array}{ccc|ccc} 3 & 0 & 4 & 1 & -3 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 4 & -3 & -6 \end{array} \right] \begin{array}{l} R1 - 4R3 \\ R2 - R3 \end{array} \sim \left[ \begin{array}{ccc|ccc} 3 & 0 & 0 & -15 & -9 & 24 \\ 0 & 2 & 0 & -4 & 4 & 6 \\ 0 & 0 & 1 & 4 & -3 & -6 \end{array} \right] \begin{array}{l} \div 3 \\ \div 2 \\ \end{array} \\
 & \sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 3 & 8 \\ 0 & 1 & 0 & -2 & 2 & 3 \\ 0 & 0 & 1 & 4 & -3 & -6 \end{array} \right] \quad A^{-1} = \begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix}
 \end{aligned}$$

c) The Cofactor Representation

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \frac{1}{-1} \begin{bmatrix} 5 & -3 & 8 \\ 2 & -2 & -3 \\ -4 & 3 & 6 \end{bmatrix} = \begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix}$$

d) Check your result by showing  $AA^{-1} = I$

$$AA^{-1} = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 3 & 1 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

4.

- i.  $\det(A) = ad - bc$
- ii.  $\det(B) = cb - ad = -(ad - bc) = -\det(A)$
- iii.  $\det(D) = d(a + kc) - c(b + kd) = ad + kdc - cb - cdk = ad - bc = \det(A)$
- iv.  $\det(C) = adk - kcb = k(ad - bc) = k \cdot \det(A)$

$A \sim B$  by a row interchange and ii shows  $\det(A) = -\det(B)$

$A \sim C$  by a row scaling and iv shows  $\det(A) = k \cdot \det(C)$

$A \sim D$  by a row interchange where a multiple of one row is added to another. iii shows that  $\det(A) = \det(D)$

5.

Forward Direction: Assume  $A_{3 \times 3}$  is such that  $\det(A) = 0$ . then the volume of the parallelepiped spanned by  $\vec{a}_1, \vec{a}_2, \vec{a}_3$  has zero volume. That is, the parallelepiped does not exist. This implies that all the vectors  $\vec{a}_1, \vec{a}_2, \vec{a}_3$  lie in the same plane, and from a linearly dependent set. Thus, by the invertible matrix theorem  $A^{-1}$  does not exist.

Backward Direction: Assume A is not invertible. Then the columns of A are linearly dependent and  $\text{span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$  is at most (in terms of dimension) a plane which has zero volume and cannot form a parallelepiped. Thus,  $\det(A) = 0$ .