

PHGN 462 Homework 6

1) (From Pollack and Stump 13.12)

This next one lets us derive the equations for group and phase velocity. We're going to start with a Gaussian wave pulse – that is, an electromagnetic wave with an overall Gaussian profile and a relatively short length. Underneath that Gaussian profile we still have the usual sinusoidal ripples at some frequency ω . Such a wave is made up of a whole bunch of different frequencies, centered on ω . We'll represent the time-domain wave amplitude as the Fourier transform of the frequency-domain wave:

$$\phi(x, t) = \int_{-\infty}^{\infty} e^{i(kx - \omega t)} f(k) \frac{dk}{2\pi}$$

where $f(k) = f_0 e^{-(k-k_0)^2 a^2}$ is a function describing the contribution to the pulse from any particular wavenumber k (which corresponds to some particular frequency). Recall that the Fourier transform of a Gaussian is a Gaussian – if it's a Gaussian pulse in the time domain, it's also a Gaussian in the frequency domain.

The Fourier transform integrates k from $(-\infty, \infty)$, but the integrand is peaked at $k = k_0$, and the width of the peak is of order $1/a$. Within a sufficiently narrow span, $\omega(k)$ will be locally linear, and can be approximated by:

$$\omega(k) = \omega(k_0) + (k - k_0)\omega'(k_0)$$

a) Evaluate the integral, and obtain explicitly the function $\phi(x, t)$. (Hint: Let $k = k_0 + q$, change the variable of integration to q , and evaluate the integral with tables or Mathematica.)

b) Show explicitly that the phase velocity is $\omega(k_0)/k_0$ and the group velocity is $\omega'(k_0)$. Keep in mind that you can find a velocity by tracking a particular value of the argument of a function.

2) Let's come up with and apply a procedure for recovering the index of refraction of a material from that material's absorption spectrum. This is the real stuff, so as with all stuff that you can actually use, there may be some annoying, fiddly bits. But it'll be worth it.

a) First off, let's assume that we have a material that is both refractive and absorptive. That means that we'll be able to write all sorts of stuff, from the permittivity to the susceptibility, as complex-valued functions of wavelength. The real parts represent the refractive behavior, and the imaginary parts represent that absorption.

Let the susceptibility of the material be:

$$\chi = \chi_1 + i\chi_2$$

Working from the definition of index of refraction and relationships between index, permittivity, and susceptibility, show that if the susceptibility is small, we can write the real part of the index of refraction as:

$$\text{Re}\{n\} \approx 1 + \frac{\chi_1}{2}$$

Keep in mind that all of these things are functions of frequency ω .

b) So how the heck do we actually measure χ_1 ? Well, we could send light of lots of different colors through the material and see how much each color refracts, thus obtaining n and χ_1 . But that's hard, and we don't usually do that. Instead we measure absorption, and use the Kramers-Kronig relations to get from absorption to what we really want.

Define the absorption function $\delta(\omega)$ to be the reciprocal of the absorption length d , which itself is defined on p. 513 of Pollack and Stump:

$$\delta(\omega) = d^{-1}$$

Working from the basic dispersion relation for plane waves, and from various identities linking the wavevector, permittivity, and susceptibility, show that:

$$\delta(\omega) \approx \frac{\omega\chi_2}{c}$$

c) Use the above results and the alternate form of the Kramers-Kronig relations (the form with the domain that goes from zero to infinity) to show that the real part of the index of refraction is related to the absorption spectrum via:

$$\text{Re}\{n(\omega)\} = 1 + \frac{c}{\pi} \int_0^{\infty} \frac{\delta(\omega')}{(\omega'^2 - \omega^2)} d\omega'$$

Pay close attention to where the omegas and omega primes are, as they contain an important bit of physics: To get the index at *any* frequency, you need to know about the absorption at *all* frequencies.

d) Now the fun part: Let's say we have a material that is strongly absorptive at one particular frequency, and not terribly absorptive anywhere else. Write down a function $\delta(\omega)$ that models that behavior. Then use the relationship from (c) to generate a plot of $\text{Re}\{n(\omega)\}$ for that material. This will almost certainly need to be done numerically. I'm only concerned with finding qualitative behavior, so feel free to set constants to one and not worry about units.

Depending on what kind of $\delta(\omega)$ you use, the integral may or may not converge cleanly. This is where it can get fiddly. Mathematica has a whole host of options for tweaking its integration routine, and the latest version is decent about recommending them to you based on the manner in which the integration is failing to converge. You can also try different $\delta(\omega)$ functions and see if some integrate better than others (hint: some do).

e) Look at your plot and tell me whether or not it has the ‘expected’ pattern of normal and anomalous dispersion. I’m actually not certain whether *all* absorption functions necessarily generate anomalous dispersion in the vicinity of an absorption peak (but they usually do), so let’s see what you get. I’m not looking for a whole lot of discussion on this part; feel free to be brief.

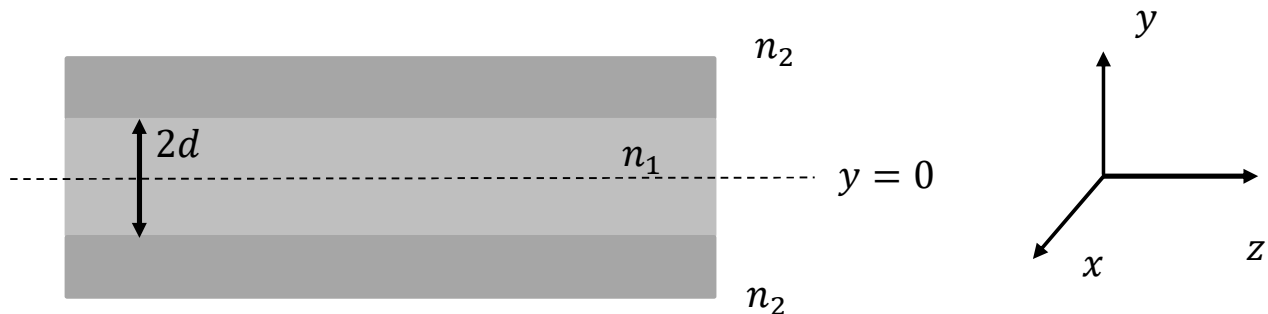
3) We’re going to be spending a lot of time talking about how E&M waves propagate in various waveguides, and not a lot of time thinking about why we bother to propagate them. Root around in books or on the interwebs and learn about the actual applications of microwave waveguides. Write up a short description of two different applications. It would be nice if at least one of them came into common usage after 1960 (a lot of microwave tech is pretty old).

And don’t take advantage of this problem by directly copying and pasting stuff from, say, the first and second websites that come up when you search for “microwave waveguides.” I know which sites those are. I also know what the third and fourth hits are, before you get any clever ideas.

4) (From Pollack and Stump 14.1)

For the TE(n) mode of propagation in the space between parallel conducting planes, show that $\vec{S}_{avg} = u_{avg} v_{gr} \hat{z}$, where u is the field energy density. Here the subscript *avg* implies an average over both t and the y coordinate. Comment on whether this helps us figure out which velocity (phase or group) is the velocity at which the mode “actually” propagates.

5) The waveguides that we’re dealing with in class all have conducting boundaries, but we have the tools we need to consider waveguides made of dielectric materials, too. These are arguably the most important kind of modern waveguide, since that category includes optical fibers. We’re going to construct TE solutions for a waveguide made up of a planar slab of dielectric with two other dielectrics above and below it. The top and bottom layers are of index n_2 and go up and down effectively forever. The inner layer is of index n_1 and is of thickness $2d$. We’ll let the dead center be $y = 0$. For simplicity, let’s say we’ll be operating at a single frequency, so that we can treat the indices of refraction as constants.



An optical fiber is basically this setup, but with a cylindrical geometry. The inner and outer layers are referred to as the core and cladding, respectively. Note that in general we have $n_1 > n_2$. The ray optics view of this system is that the light propagates down the center, totally internally reflecting off of the two interfaces – and total internal reflection requires that you be going from high index into low index.

Anyway, this is an awful lot like a parallel plate waveguide (which we know how to deal with), except that the top and bottom regions aren't conducting, so the fields in those regions aren't forced to be exactly zero. However, if the radiation is bouncing along by way of total internal reflection, the fields in the top and bottom regions are quite likely to be evanescent (which we also know how to deal with).

Our starting points include the basic form of an evanescent E-field and the basic form of TE modes in a parallel-plate conductor:

$$\vec{E}_{evan} = E_0 e^{-k_y y} e^{i(k_z z - \omega t)} \hat{i} \quad \vec{E}_{TE} = E_0 \sin k_y y e^{i(k_z z - \omega t)} \hat{i}$$

We're going to use these as trial functions to build up solutions for the dielectric slab waveguide.

a) Write the trial E-fields for each of the top, middle, and bottom regions. You'll have to slightly adapt the evanescent field equation above to account for the fact that the evanescent fields begin at $y = \pm d$ and need to decay as we go farther up/down. Then use Faraday's law to find the corresponding B-fields in those regions. Make no assumptions beyond the basic form of the fields. In other words, don't assume that the k 's, E_0 's, or ω 's in different regions are the same.

b) Apply our four basic boundary conditions on E and B to these fields. Figure out which of the k 's, E_0 's, or ω 's are the same in each region, and show that k_y and k'_y are related by:

$$-\cot k_y d = \frac{k'_y}{k_y}$$

Where k_y and k'_y are the y components of the wavevector in media 1 and 2, respectively.

c) Use the fact that the E-fields in regions 1 and 2 must satisfy the wave equation to show that:

$$k_y^2 + k'_y{}^2 = \omega^2(\epsilon_1 - \epsilon_2)\mu_0$$

and then combine that with the result from (b) to show that the dispersion relation for this waveguide is:

$$-\cot k_y d = \sqrt{\frac{\omega^2(\epsilon_1 - \epsilon_2)\mu_0}{k_y^2} - 1}$$

d) At this point we have all the relations we need to characterize any particular mode, or to find out how many modes a particular waveguide will support. Let's suppose we have a glass waveguide ($n_1 = 1.5$) with plastic cladding ($n_2 = 1.3$) and a d of 3 microns. Let's also suppose we inject green laser light with a vacuum wavelength of 532 nm. How many modes can exist in our waveguide? Feel free to solve the dispersion relation numerically or graphically, but show your work either way.