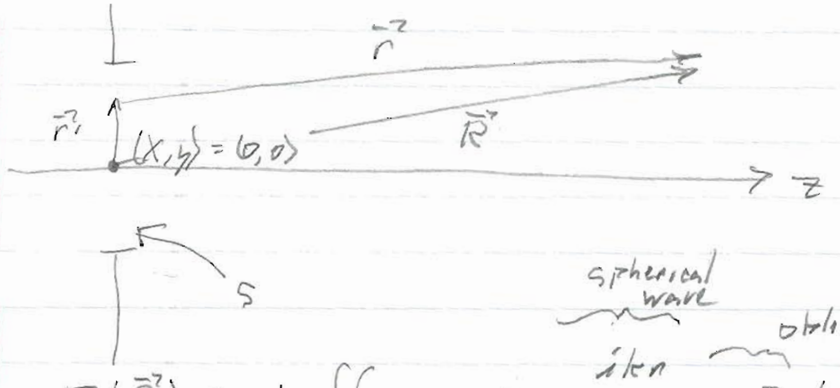
Diffraction and the angular spectrum



scalar field
$r >> \lambda$

$$E(\vec{R}) = \frac{1}{i\lambda}\iint E(r')\underbrace{\frac{e^{ikr}}{r}}_{\text{spherical wave}}\underbrace{\cos\theta}_{\text{obliquity factor.}}\,ds$$

$$U(x,y,z) = \frac{1}{i\lambda}\iint U(x,y,0)\frac{e^{ik|\vec{R}-\vec{r}'|}}{|\vec{R}-\vec{r}'|}\frac{z}{|\vec{R}-\vec{r}'|}\,dx\,dy$$

$$r = |\vec{R}-\vec{r}'| = \sqrt{(x-x')^2+(y-y')^2+z^2}$$

Symbolically,

$$U(x,y,z) \sim U(x,y,0) \otimes g(x,y,z)$$

This suggests treatment with linear systems theory
- superposition? ✓ (linear)
- shift invariant ✓

can we write in transform space?

$$A(f_x,f_y,z) \sim A(f_x,f_y,0)\,G(f_x,f_y,z)$$

Angular spectrum:

$$A(f_x,f_y;z=0) = \iint U(x,y;0)\,e^{-i2\pi(f_x x+f_y y)}\,dx\,dy$$

$$= \mathcal{F}_{xy}\{U(x,y,0)\}$$

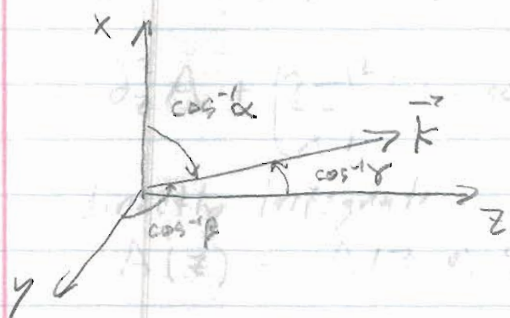
example: plane wave

$$p(x,y,z;t) = e^{i(\vec{k}\cdot\vec{r} - \omega t)} = e^{i(k_x x + k_y y + k_z z - \omega t)}$$

write $\vec{k} = \frac{2\pi}{\lambda}(\alpha \hat{x} + \beta \hat{y} + \gamma \hat{z})$ $\quad \lambda = \lambda_0/n$ inside medium.

$$\alpha = \hat{x}\cdot\frac{\vec{k}}{k} = \cos\theta_{xk} = \frac{k_x}{k} = \sin\theta_x$$

α, β, γ = "direction cosines"



$$\frac{\vec{k}\cdot\vec{k}}{n^2 k_0^2} = 1 \rightarrow \alpha^2 + \beta^2 + \gamma^2 = 1$$

put time dependence into amplitude

$$P(x,y,z) = e^{ink_0\gamma z} e^{ink_0(\alpha x + \beta y)}$$

$$\gamma = \sqrt{1 - \alpha^2 - \beta^2}$$

calculate spectrum:

$$A_p(f_x, f_y; z) = e^{ink_0 z} \tilde{f}_{xy}\left\{ e^{ink_0(\alpha x + \beta y)} \right\}$$

$$\tilde{f}_x\left\{ e^{\pm i 2\pi f_{x_0} x} \right\} = \delta\{f_x \pm f_{x_0}\}$$

$$f_{x_0} = \frac{nk_0\alpha}{2\pi} = \frac{\alpha}{\lambda} \qquad \lambda = \lambda_0/n$$

$$f_{y_0} = \beta/\lambda$$

angular spectrum:

$$A_p(f_x, f_y; z) = e^{ink_0 z} \delta\left(f_x - \frac{\alpha}{\lambda}\right)\delta\left(f_y - \beta/\lambda\right)$$

Propagation of the angular spectrum

$$U(x,y,z) = \mathcal{F}^{-1}\{A(f_x,f_y,z)\} = \iint A(f_x,f_y,z)\, e^{+i2\pi(f_x x + f_y y)}\, df_x\, df_y$$

$\rightarrow$ Helmholtz (wave) eqn:

$$\nabla^2 U + k^2 U = 0$$

$$\partial_x U = \partial_x \mathcal{F}^{-1}\{A\} = i2\pi f_x \mathcal{F}^{-1}\{A\} \qquad f_x = \alpha/\lambda$$

$$\rightarrow \partial_z^2 A + \left(\frac{2\pi}{\lambda}\right)^2 (1-\alpha^2-\beta^2) A = 0$$

directly integrate

$$A(z) = A(z=0)\, e^{i\frac{2\pi}{\lambda}(1-\alpha^2-\beta^2)^{1/2} z} = A_0 e^{ik_z z}$$

propagation in f_x, f_y space is simple:

1) calculate at $z=0$
$$A(f_x,f_y,0) = \mathcal{F}\{U(x,y,0)\}$$

2) each component propagates as
$$A(f_x,f_y,0)\exp(i k_z(f_x,f_y) z)$$

3) transform back.

Note: if α, β are too large, $k_z = k_0 n\sqrt{1-\alpha^2-\beta^2}$
$\rightarrow$ imaginary

waves $\rightarrow$ evanescent, don't propagate.
$\therefore$ impose bandwidth limit $\text{circ}\left((\alpha^2+\beta^2)^{1/2}\right)$

Transfer function $G(f_x,f_y) = e^{i\frac{2\pi}{\lambda}(f_x^2+f_y^2)^{1/2}} \quad \text{circ}\left(\lambda\sqrt{f_x^2+f_y^2}\right)$

Can do <u>non-paraxial</u> propagation with this!

Paraxial Propagation

- for small angles, we can approximate $\sqrt{\ }$ in propagator. $\sqrt{\alpha^2+\beta^2}$
- equivalent to approx $r = \sqrt{(x-x')^2+(y-y')^2+z^2}$

back to diffraction integral:

$$U(x,y,z) = \frac{1}{i\lambda}\iint U(x',y')\,\frac{e^{ikr}}{r}\cos\theta\,dx'dy'$$

paraxial approximation

amplitude term $\frac{\cos\theta}{r} = \frac{z}{r^2} \approx \frac{1}{z}$

phase: $kr = \frac{2\pi n}{\lambda_0}\, z\sqrt{1+\left(\frac{x-x'}{z}\right)^2+\left(\frac{y-y'}{z}\right)^2}$

$$\approx kz\left(1+\frac{(x-x')^2}{2z^2}+\frac{(y-y')^2}{2z^2}\right)$$

$$= k\left(z+\frac{x^2+y^2}{2z}-\frac{xx'+yy'}{z}+\frac{x'^2+y'^2}{2z}\right)$$

note $\frac{x}{z} \approx \sin\theta_x$ so $\frac{x}{z}\frac{1}{\lambda} \approx f_x$

$$U(x,y,z) \doteq \frac{1}{i\lambda z}\, e^{ikz}\, e^{ik(x^2+y^2)/2z}\iint U(x',y')\, e^{i\frac{k}{2z}(x'^2+y'^2)}\, e^{-i2\pi(f_x x'+f_y y')}\, dx'dy'$$

$$= \frac{1}{i\lambda z}\, e^{ikz}\, e^{ik(\ \ )/2z}\, \mathcal{F}_{x',y'}\left\{U(x',y')\, e^{ik(x'^2+y'^2)/2z}\right\}$$

= Fresnel propagation

Fraunhofer propagation: drop quadratic phase factor.

→ Fourier transform!

when is $e^{ik(x'^2+y'^2)/2z} \approx 1$?

Far-field approximation

when is $e^{ik(x'^2+y'^2)/2z} \approx 1$?

let input be bounded by $x', y' < a$, $z \to L$

$$e^{ika^2/2L} \approx 1 \text{ if } \frac{a^2}{L\lambda} << 1$$

$F = \frac{a^2}{L\lambda}$ = Fresnel #

$F << 1 \rightarrow$ Fraunhofer

recall for Gaussian beams:

$$z_R = \frac{\pi W_0^2}{\lambda} \qquad \text{so} \qquad \frac{\pi a^2/\lambda}{L} \sim \frac{z_R}{L\pi}$$

Fraunhofer diffraction

From paraxial analysis

$$U(x,y,z) = \frac{e^{ikz}}{i\lambda z} e^{ik\frac{(x^2+y^2)}{2z}} \iint U(x',y') e^{\frac{ik}{2z}(x'^2+y'^2)} e^{-i2\pi(f_x x + f_y y)} dx'dy'$$

if input is spatially localized to a radius a, then we can drop quadratic phase term inside integral:

$$\max\left(\frac{k(x'^2+y'^2)}{2z}\right) \sim \frac{ka^2}{2z}$$

$$\text{if } \frac{ka^2}{2z} \ll \pi \text{ or } \underbrace{F = \frac{a^2}{z\lambda}}_{\text{Fresnel number.}} \ll 1$$

dropping quadratic phase → Fraunhofer diffraction
→ Fourier transform!

discuss terms in expression:

1) $$\iint U(x',y') e^{-i2\pi(f_x x + f_y y)} dx'dy' = \tilde{f}_{x'} \tilde{f}_{y'} \{U(x',y')\}$$

$$f_x = \frac{x}{z}\frac{1}{\lambda} \doteq \frac{\sin\theta_x}{\lambda} \qquad f_y = \frac{y}{z}\frac{1}{\lambda} \doteq \frac{\sin\theta_y}{\lambda}$$

do FT w.r.t. x', y' → $U(f_x, f_y)$ → $U(x,y,z)$

note wavelength + z scaling of pattern.

λ is inside medium $= \lambda_0/n$

2. $e^{i\frac{k}{2z}(x^2+y^2)}$ = spherical wavefront
FT of input field will have phase that will modify this

3. e^{ikz} = overall propagation phase.

4. $\frac{1}{i\lambda z}$ → intensity varies as $\frac{1}{\lambda^2 z^2}$
more diffraction with long λ → bigger spread.
→ less intensity.

Fraunhofer diffraction examples.

$$U(x,y,z) = \frac{e^{ikz}}{i\lambda z} e^{i\frac{k}{2z}(x^2+y^2)} \underbrace{\mathcal{F}_{x'y'}\{U(x',y')\}}_{A(f_x,f_y)}$$

$f_x = \frac{x}{z\lambda}$

$f_y = \frac{y}{z\lambda}$

$U(x',y')$ is field <u>leaving</u> plane $z=0$

$$\underbrace{U_t(x',y')}_{\text{transmitted field}} = \underbrace{U_i(x,y,0)}_{\text{input field (complex)}} \cdot \underbrace{t_A(x,y)}_{\text{aperture plane transmission fcn. (complex)}}$$

note:

$$A_t(f_x,f_y) = A_i(f_x,f_y) \otimes T(f_x,f_y)$$

1) rectangular aperture, uniform plane wave illumination
normal incidence

$$U_t(x',y') = \underbrace{E_0}_{U_i} \underbrace{\mathrm{rect}(x/a)\,\mathrm{rect}(y/b)}_{t_A}$$

$$A(f_x,f_y) = ab\,\mathrm{sinc}(\pi f_x a)\,\mathrm{sinc}(\pi f_y b)$$

$$U(x,y,z) = E_0 \frac{e^{ikz}}{i\lambda z} e^{i\frac{k}{2z}(x^2+y^2)} \cdot ab\,\mathrm{sinc}\left(\frac{\pi a}{z\lambda}x\right)\mathrm{sinc}\left(\frac{\pi b}{z\lambda}y\right)$$

full width: 1st zero at $x = \frac{\lambda z}{a}$ where $\sin(\pi) = 0$

$FW = 2\lambda z/a$ diffraction ½ angle $\theta \sim \frac{x_{FW}}{z}$

$\underline{\theta \sim \lambda/a}$

$$I(x,y,z) \propto \frac{E_0^2}{\lambda^2 z^2} a^2 b^2\,\mathrm{sinc}^2(\quad)\,\mathrm{sinc}^2(\quad)$$

2) plane wave input, angle of incidence θ_{x0}

$$U_i = E_0 e^{i k_0 (\sin\theta_{x0}) x'}$$

by shift theorem, $\mathcal{F}\{e^{i 2\pi f_{x0} x} g(x)\} = G(f_x - f_{x0})$

identify $2\pi f_{x0} = k_0 \sin\theta_{x0} \rightarrow f_{x0} = \frac{\sin\theta_{x0}}{\lambda}$

since $f_x = \frac{x}{\lambda z} \rightarrow \operatorname{sinc}\left(\pi (f_x - f_{x0}) a\right)$

$$= \operatorname{sinc}\underbrace{\left(\frac{\pi a}{z\lambda} x - \frac{\sin\theta_{x0} \cdot \pi a}{\lambda}\right)}_{\frac{\pi a}{\lambda z}(x - z\sin\theta_{x0})}$$

geometrically, expect shift by $z \tan\theta_{x0} = z \frac{\sin\theta_{x0}}{\cancel{\cos\theta_{x0}}}$
small angle

3) double slit, width a, separation d
normal incidence

$$t_A(x,y) = \left[\operatorname{rect}\left(\frac{x - d/2}{a}\right) + \operatorname{rect}\left(\frac{x + d/2}{a}\right)\right] \operatorname{rect}(y/b)$$

shift $\quad A(f_x) = a e^{-2\pi f_x \left(\frac{d}{2}\right)} \operatorname{sinc}(\pi f_x a) + a e^{+2\pi f_x \frac{d}{2}} \operatorname{sinc}(\pi f_x a)$

$$= 2a \cos(\pi f_x d) \operatorname{sinc}(\pi f_x a)$$

conv: $\quad t_A(x,y) = \operatorname{rect}\left(\frac{x}{a}\right) \otimes \left(\delta(x - d/2) + \delta(x + d/2)\right)$

$$A(f_x) = a \operatorname{sinc}(\pi f_x a) \cdot \underbrace{\left(e^{i 2\pi f_x d/2} + e^{-i 2\pi f_x d/2}\right)}_{2\cos(\pi f_x d)}$$

fringe spacing: $\pi \frac{x_f}{\lambda z} d = 2\pi \qquad x_f = \frac{2\lambda z}{d}$

4) transmission spatial light modulator SLM

program $n(x)$

modulation $e^{i \frac{2\pi}{\lambda_0} n(x) \delta z}$

linear phase ramp:

$$n(x) = n_{max} X / X_0$$

$$t_A(x) = e^{\overbrace{i \left(\frac{2\pi n_{max}}{\lambda_0 X_0} \delta z\right) x}^{}}$$

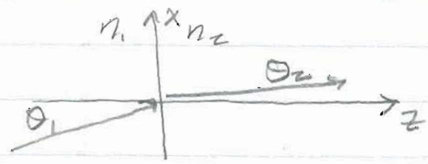
$$\underbrace{\quad\quad}_{2\pi f_{x0} X} \qquad f_{x0} = \frac{n_{max} \delta z}{\lambda_0 X_0} = \frac{\sin\theta_{x0}}{\lambda_0}$$

steers beam by θ_{x0}

note: typically we can get only few $\cdot \pi$ delay.

$\therefore$ wrap every 2π

5) transition from n_1 to n_2, surface normal $\parallel z$



incident wave $E_0 e^{i k_x X} = E_0 e^{i k_0 n_1 \sin\theta_1 x}$

incident angular spect: $E_0 \delta\left(f_x - \frac{k_0 n_1 \sin\theta_1}{2\pi}\right) \delta(f_y)$

where $f_x = \alpha/\lambda$ λ is inside medium $= \lambda_0 / n_1$

starting $f_{x_1} = \frac{n_1 \sin\theta_1}{\lambda_0}$

by Snell $f_{x_2} = \frac{n_2 \sin\theta_2}{\lambda_0} = f_{x_1}$

$\therefore$ $A(f_x, f_y)$ is unchanged when crossing surface $\perp z$

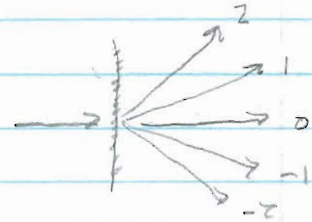
But an angled surface $\rightarrow$ change in direction, shift in f_x

e.g. thin wedge $d(x) = (X - X_0) \cdot \gamma$

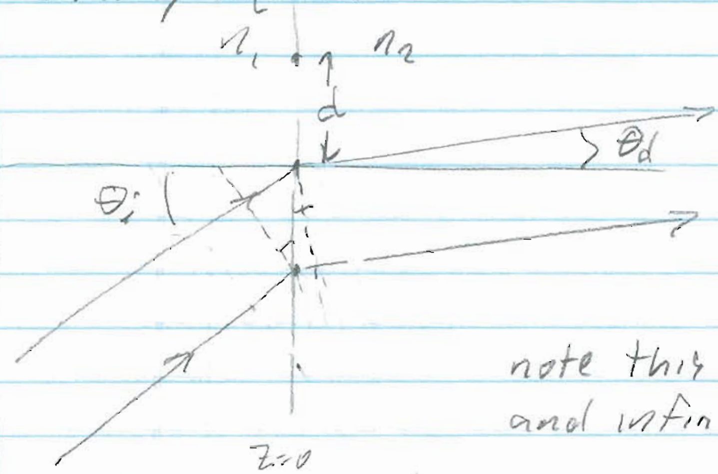
$\rightarrow$ phase $\frac{2\pi}{\lambda_0} n\, d(x) = \frac{2\pi n (X - X_0)\gamma}{\lambda_0} \Rightarrow f_{x0} = \frac{n\gamma}{\lambda_0}$ as in SLM example.

Diffraction gratings

- used for angular dispersion
- transmission
- reflection



Grating equation



$$\Delta(n\ell) = n_1 d \sin\theta_i - n_2 d \sin\theta_d$$
$$= m\lambda$$

$$\rightarrow n_2 \sin\theta_d = \frac{m\lambda_0}{d} + n_1 \sin\theta_i$$

note this assumes point sources and infinite plane waves

in spatial frequency variables:

$$f_{xd} = \frac{m}{d} + f_{xi}$$

in k_x's $\quad k_{xd} = k_g + k_{xi}$ $\quad$ grating has a k contrib.

reflection from facet - normal incidence

change direction by 2α

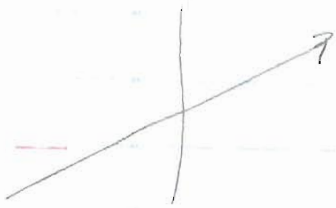
$$\rightarrow e^{ik \sin 2\alpha\, x}$$

facet width a $\qquad \mathrm{rect}(x/a)$

infinite array, spacing d $\qquad \mathrm{comb}(x/d)$

overall edge of grating $\qquad \mathrm{rect}(x/D)$

blazed diffraction grating, reflection



transmission

reflection.

single element:

phase from tilt: $e^{i k_0 x \sin\alpha) \cdot 2}$

look at relative phase.

$\therefore$ each element is $\text{rect}(x/a)\, e^{i k_0 \cdot 2 \sin\alpha\, x}$

array: $\text{rect}(x/D)\ \text{comb}(x/d)$

full grating: $\left[\text{rect}(x/D)\ \text{comb}(x/d)\right] \otimes \left[\text{rect}(x/a)\ e^{i(k_0 2 \sin\alpha) x}\right.$

groove: $\rightarrow\ a\, \text{sinc}\left(\pi a \left(f_x - \frac{2 k_0 \sin\alpha}{2\pi}\right)\right)$

$$= a\, \text{sinc}\left(\pi a \left(\frac{x}{\lambda z} - \frac{2 \sin\alpha}{\lambda}\right)\right)$$

array: $D\, \text{sinc}(\pi f_x D) \otimes \frac{1}{d}\, \text{comb}(f_x d)$

full grating: product

$$\left[D\, \text{sinc}(\pi f_x D) \otimes \frac{1}{d} \text{comb}(f_x d)\right] a\, \text{sinc}\left(\frac{\pi a}{\lambda z}(x - 2z \sin\alpha)\right)$$

interpretation:

smallest feature $\text{sinc}\left(\frac{\pi D}{\lambda z} x\right) \rightarrow$ lineshape

width determined by size of grating.

$D = Nd$

diffraction orders

$$\text{comb}(f_x d) \rightarrow \sum \delta(f_x - m/d)$$

peaks at $\frac{x}{\lambda z} = m/d$

$x/z \sim \sin\theta_d \quad \rightarrow m\lambda = d\sin\theta_d$

$m = 0, \pm 1, \pm 2, \ldots$

groove shape → envelope on orders.

if $\alpha = 0$, max at $x = 0$, zero order.

can adjust peak:

$x_{pk} = 2z\sin\alpha$

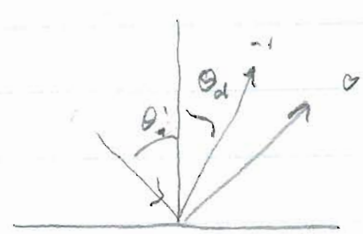
for first order, $\rightarrow 2\sin\alpha = \sin\theta_d$

Non-normal incidence

$$E_{in} = E_0 e^{ik_0 \sin\theta_i \cdot x}$$

shift all by $x' = z\sin\theta_i$

$$\rightarrow m\lambda = d(\sin\theta_d - \sin\theta_i)$$



Littrow angle : $\theta_d = -\theta_i$, $m = -1$

"blaze wavelength" peak into -1 order at Littrow for λ_{blaze}