

Waves in magnetized plasmas

→ Faraday notation: induced optical activity

Whistler, Alfvén waves

2-fluid plasma equations (cold $T=0$, no thermal pressure)
continuity $\alpha = e$ electrons $\alpha = i$ ions $n_e \vec{V}_e = \vec{J}_e$
 $\partial_t n_\alpha + \nabla \cdot n_\alpha \vec{V}_\alpha = 0$

momentum:

$$n_\alpha m_\alpha (\partial_t + \vec{V}_\alpha \cdot \nabla) \vec{V}_\alpha = n_\alpha q_\alpha \left(\vec{E} + \frac{1}{c} \vec{V}_\alpha \times \vec{B} \right)$$

Newton: total/connective derivative. Lorentz

Maxwell:

$$\nabla \cdot \vec{E} = \sum 4\pi n_\alpha q_\alpha$$

$$\nabla \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c} \sum n_\alpha q_\alpha \vec{V}_\alpha$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

These are nonlinear equations.

Linearize to find wave solutions:

$$n_\alpha \rightarrow n_{\alpha 0} + n_{\alpha 1} e^{-i\omega t} \quad n_{\alpha 1} \ll n_{\alpha 0}$$

$$\vec{E} \rightarrow \vec{E}_1 e^{-i\omega t}$$

$$\vec{B} \rightarrow \vec{B}_0 + \vec{B}_1 e^{-i\omega t} \quad \vec{B}_0 = B_0 \hat{z} \text{ steady field}$$

$$\vec{V}_\alpha \rightarrow \vec{V}_{\alpha 1} \quad \vec{B}_1 \ll \vec{B}_0 \text{ wave ampl.}$$

keep terms to 1st order in perturbed quants.
continuity:

$$-i\omega n_{\alpha 1} + n_{\alpha 0} \nabla \cdot \vec{V}_{\alpha 1} = 0$$

momentum:

$$-i\omega \vec{V}_{\alpha 1} = \frac{q_\alpha}{m_\alpha} \left(\vec{E}_1 + \frac{1}{c} \vec{V}_{\alpha 1} \times \vec{B}_0 \right)$$

Maxwell eqns:

$$\nabla \times \vec{E}_1 = \frac{i\omega}{c} \vec{B}_1$$

$$\nabla \times \vec{B}_1 = -i\omega \vec{E}_1 + \frac{4\pi}{c} \sum_{\alpha} q_{\alpha} n_{\alpha} \vec{V}_{\alpha}$$

As in HW1 the charged particles will move in orbits around the magnetic field: cyclotron motion.

- our eqns are for the fluid, but momentum eqn is the same.

Solve for velocity

$$V_{x1z} = \frac{i q_e}{m_e \omega} E_{1z}$$

$$V_{x1x} = \frac{i q_e}{m_e \omega} \left(E_{1x} + \frac{1}{c} V_{x1y} B_0 \right) =$$

$$V_{x1y} = \frac{i q_e}{m_e \omega} \left(E_{1y} - \frac{1}{c} V_{x1x} B_0 \right)$$

$$\omega_p^2 = \frac{4\pi n_{\alpha 0} q_{\alpha}^2}{m_{\alpha}} \quad \omega_{ce} = \left| \frac{q_e B_0}{m_e c} \right|$$

$$4\pi n_{\alpha 0} q_{\alpha} V_{x1x} = i \frac{\omega^2}{\omega} \left(E_{1x} + \omega_{ce} V_{x1y} \right)$$

$$4\pi n_{\alpha 0} q_{\alpha} V_{x1y} = i \frac{\omega^2}{\omega} \left(E_{1y} - \omega_{ce} V_{x1x} \right)$$

solve for V_{x1x} , V_{x1y} in terms of $\vec{E}_1 \rightarrow \nabla \times \vec{B}_1$

$$\rightarrow \nabla \times \vec{B}_1 = -i \frac{\omega}{c} \vec{\epsilon} \cdot \vec{E}_1$$

with $\vec{\epsilon} = \begin{bmatrix} \epsilon_1 & -i\epsilon_2 & 0 \\ i\epsilon_2 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix}$ type in 4.9.6

$$\epsilon_1 = 1 + \sum_{\alpha} \frac{\omega_p^2}{\omega_{ce}^2 - \omega^2}$$

$$\epsilon_2 = 1 - \sum_{\alpha} \frac{\omega_p^2}{\omega_{ce}^2 - \omega^2}$$

$$\epsilon_3 = \frac{\omega_{ce} \omega_p^2}{\omega (\omega_{ce}^2 - \omega^2)} - \frac{\omega_{ci}}{\omega} \frac{\omega_p^2}{\omega_{ci}^2 - \omega^2}$$

this leads to the wave equation

$$\nabla \times (\nabla \times \vec{E}_1) = \frac{\omega^2}{c^2} \vec{\epsilon} \cdot \vec{E}_1$$

and, as usual, we look for plane wave solutions

- solve for general direction is complicated, as in birefringent media.
- choose natural directions

For $B_0 \rightarrow 0$ $\omega_m \rightarrow 0$ $\epsilon_2 \rightarrow 0$ $\epsilon_1 = \epsilon_3$ diagonal

$$\epsilon = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_i^2}{\omega^2} \quad \text{plasma dispersion}$$

$$\omega_p^2 \gg \omega_i^2 \quad \text{b/c of masses}$$

$$n = \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2} \quad \text{typically drop ion term.}$$

high ω $\omega > \omega_p$ wave propagates n is real

low ω $\omega < \omega_p$ $n = \text{pure imag.} \rightarrow \text{damped, (not absorbed)}$
 $\rightarrow \text{reflection.}$

$$B_0 \neq 0 \quad \vec{k} = k \hat{z}$$

check sign.

$$-\vec{k} \times (\vec{k} \times \vec{E}_1) = \vec{k} \vec{E}_1 - \vec{k} (\vec{k} \cdot \vec{E}_1) = \frac{\omega^2}{c^2} \vec{\epsilon} \cdot \vec{E}_1$$

is $\vec{k} \cdot \vec{E}_1 = 0$? $i \vec{k} \cdot \vec{E}_1 = 4\pi \sum_a n_{a1} q_a$
 $\rightarrow i k E_{1z}$

$$\vec{k} \times \vec{E}_1 = \frac{\omega}{c} \vec{B}_1$$

$\rightarrow \vec{B}_1$ is in $x-y$ plane.

$$i \vec{k} \times \vec{B}_1 = -i \frac{\omega}{c} \vec{E}_1 + \frac{4\pi}{c} \sum_a q_a n_a \vec{V}_i \rightarrow \vec{E}_1, \vec{V}_i \text{ in } x-y \text{ plane.}$$

$$k^2 \vec{E}_i = \frac{\omega^2}{c^2} \vec{E} \cdot \vec{E}_i$$

this looks like:

$$\frac{\omega^2}{c^2} \begin{pmatrix} \epsilon_1 - i\epsilon_2 & & \\ & \epsilon_1 + i\epsilon_2 & \\ & & \epsilon_2 \end{pmatrix} \begin{pmatrix} E_{ix} \\ E_{iy} \\ 0 \end{pmatrix} = k^2 \begin{pmatrix} E_{ix} \\ E_{iy} \\ 0 \end{pmatrix}$$

sign type in
eqn 4.9.6

which is an eigenvalue eqn.

we can reduce the dimensions to 2

$$\begin{pmatrix} \epsilon_1 - n^2 & i\epsilon_2 \\ -i\epsilon_2 & \epsilon_1 - n^2 \end{pmatrix} \begin{pmatrix} E_{ix} \\ E_{iy} \end{pmatrix} = 0$$

eigenvalues

$$(\epsilon_1 - n^2)^2 - \epsilon_2^2 = 0$$

$$n^2 = \epsilon_1 \pm \epsilon_2$$

eigenvectors

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix} \quad \text{right and left circular. (plasma convention)}$$

$$n_L = \sqrt{\epsilon_1 + \epsilon_2} \quad n_R = \sqrt{\epsilon_1 - \epsilon_2}$$

By adding B_0 we have induced optical activity.

→ rotation of linear pol. thru medium

= Faraday notation.

Note R,L convention in book is opposite from traditional optics.

for fixed ions (drop ion terms)

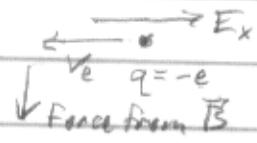
$$n_L = \left(1 + \frac{\omega_p^2}{\omega_{ce}^2 - \omega^2} \left(1 \mp \frac{\omega_{ce}}{\omega} \right) \right)^{1/2} = \left(1 - \frac{\omega_p^2}{\omega_{ce}^2 + \omega^2} \right)^{1/2}$$

for $\omega_{ce} \leq \omega \leq \omega_p$ no waves (stop band)

for $\omega < \omega_{ce}$ → only one handedness.

For RHP (with respect to \vec{B}_0), E-field drives cyclotron motion

charge -e accelerates from E_x



since $\vec{B} = B_0 \hat{z}$ $-e \frac{v_e}{c} \times \vec{B}$ is $-\hat{y}$
electron moves in RH direction around \vec{B}

when $\omega = \omega_{ce}$ → resonance for RHP, not LHP

Whistler waves:

for low freq. waves $\omega < \omega_{ce}$, plasma transmits only RMP. and is very dispersive.

→ impulse → very chirped.

high ω comes out first

