

## Gaussian beam propagation

Qualitative analysis: wave eqn:  $\nabla^2 \psi + \frac{n^2 c^2}{c^2} \psi = 0$

plane waves are one solution to the wave equation

$$\psi(\vec{r}, t) = \psi_0 e^{i(k_r \cdot \vec{r} - \omega t)}$$

what are other solutions?

- assume  $n(\vec{r})$ , no boundaries

- choose geometry: cylindrical  $\nabla_r^2 \rightarrow \nabla_r^2 + \partial_\theta^2$   
spherical

spherical wave:

$$\psi \propto \frac{1}{r} e^{i(kr - \omega t)}$$

↓  
scalar  $r$

+ outward  
- inward.

$$\text{intensity} \propto \frac{1}{r^2}$$

solution only away from  $r=0$

cylindrical wave:

$$\psi \propto \frac{1}{\sqrt{r}} e^{i(kr - \omega t)}$$

$$\text{intensity} \propto \frac{1}{r}$$

this is large  $kr$  limit of the Bessel  $J_0(z)$   
can combine w/  $e^{ikz} \rightarrow$  conical wave

How to make these waves?

spherical: point source (ideal)

dipole radiation  $\sin \theta e^{ikr}$

lens 

cylindrical: waveguide leakage



"axicon" lens (cone shape)



Paraxial spherical waves:

- assume wave is a segment of a spherical wave

- prop. in  $z$  direction,  $z \gg x, y$

$$e^{ikr} \rightarrow kr = k z \left(1 + \frac{x^2+y^2}{z^2}\right)^{1/2}$$

$$\approx k z \left(1 + \frac{1}{2} \frac{x^2+y^2}{z^2}\right) \text{ to 1st order.}$$

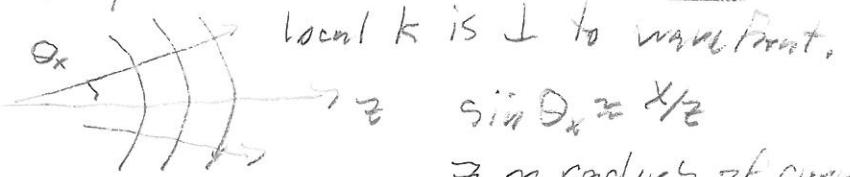
$$\rightarrow e^{ikz} e^{ik\left(\frac{x^2+y^2}{2z}\right)} \text{ diverging paraxial wave}$$

At  $x=y=0$ ,  $\rightarrow e^{ikz}$  as for plane wave.

At fixed  $z$ , phase is quadratic in  $x, y$ .

- wavefront  $\frac{k(x^2+y^2)}{2z} - \text{wt constant.}$

as  $x, y \gg z$  to maintain phase,  
i.e. outer parts of beam arrive later



$z$  n radius of curvature

In Gaussian beams: wave is paraxial, will have same form.

## Beam propagation - diffraction integrals

Hyugens principle:

represent a plane wave as sum over spherical waves:



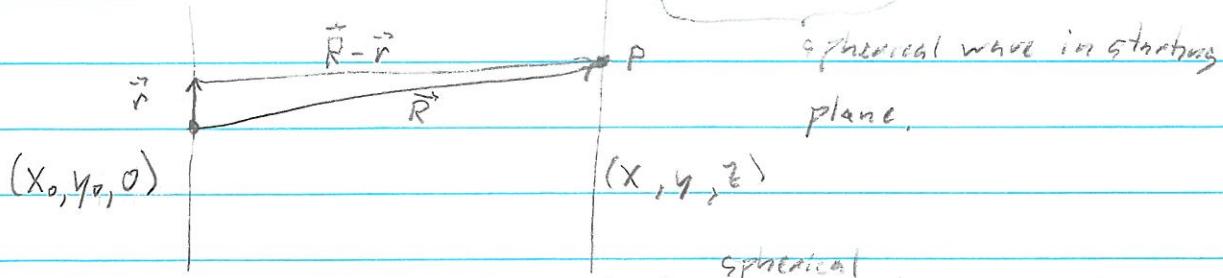
- those could be real sources (dipole antennas)
- scattering points (atoms in a lattice)

- here, we treat it as a mathematical construction,
- in more modern language these wavelets are the Green's func for the wave eqn.

\* in these notes:  $e^{-ik\vec{z} + iwt}$  convention to conform to Svelto, Siegman.

Integral Form:

$$U_{\text{diff}}(\vec{R}) \propto \left\{ U_{\text{tr}}(\vec{r}) \frac{e^{-ik|\vec{R}-\vec{r}|}}{|\vec{R}-\vec{r}|} dS \right\}$$



- there are some corrections to the waves to ensure they propagate in the forward direction. These get dropped when we look in paraxial appx.

Approximations: paraxial  $\rightarrow$  Fresnel  $\rightarrow$  Fraunhofer.

- assume some max extent of field in x, y plane  $x, y < a$

- apertures or beam.

$$\text{approx } |\vec{R}-\vec{r}| = \sqrt{(x-x_0)^2 + (y-y_0)^2 + z^2}$$

$$= z \sqrt{1 + \frac{x_0^2 + y_0^2}{z^2} - 2 \frac{xx_0 + yy_0}{z^2}}$$

for  $a \ll z$  so that  $\frac{x_0^2 + y_0^2}{z} \ll 1$

and  $\frac{xx_0 + yy_0}{z} \ll 1$  i.e. paraxial

$\frac{1}{|\vec{R} - \vec{r}|} \approx \frac{1}{z}$  amplitude is not as sensitive to change as the phase.

$$-ik|\vec{R} - \vec{r}| \rightarrow -ikz - ik \left( \frac{x_0^2 + y_0^2}{2z} - \frac{xx_0 + yy_0}{z} \right)$$

→ Fresnel integral (near field):

$$U(x, y, z) = \frac{i e^{-ikz}}{\lambda z} \iint_{-\infty}^{\infty} U(x, y) e^{\frac{-ik(x^2 + y^2)}{2z} + ik\left(\frac{x}{z}x_0 + \frac{y}{z}y_0\right)} dx_0 dy_0$$

- if  $\frac{ka^2}{2z} = \frac{\pi a^2}{\lambda z} \ll 1$  deep quasi-phase → Fraunhofer "far field"

- in this case far field is a 2-D Fourier transform:

$$\frac{kx}{z} \sim k \sin \theta_x \equiv \beta_x = \text{spatial frequency variable}$$

Notes:

- $U(x, y)$  is scalar field that leaves  $x, y$  plane

- if an actual aperture is present,  $A(x, y)$  then

$$U(x, y) = E(x, y) A(x, y)$$

- Fresnel works in all cases where Fraunhofer does.