

# Nonlinear Fiber Optics

Guest Lecturer

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# Nonlinear Pulse Propagation in Fibers

## And Maybe Some Other Stuff

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Daniel Adams, PhD

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This is your  
captain speaking.  
Next stop is  
whenever.  
Just be like,  
“stop.”



# In the News

## Full characterization of polarization states of light via direct measurement

Jeff Z. Salvail, Megan Agnew, Allan S. Johnson, Eliot Bolduc, Jonathan Leach & Robert W. Boyd

Salvail, J. Z., Agnew, M., Johnson, A. S., Bolduc, E., Leach, J., & Boyd, R. W. (2013). Full characterization of polarization states of light via direct measurement. Nat Photon,

# A Note on Differential Equations

- Fundamental way to define functions
  - Other fundamental way?
- Solutions provide insight into physical systems
- Ways to derive differential Eqns?

# The Nonlinear Wave Equation

- Where do we get the wave Equation?

$$\nabla \cdot D = \rho_f$$

$$\nabla \cdot B = 0$$

$$\nabla \times H = J + \partial_t D$$

$$\nabla \times E = -\partial_t B$$

- Maxwell's Equations?

# Fine... Where do we get Maxwell's Equations?

- The Easy way... Considering charges and thinking and stuff...
- Special Relativity and the Coulomb Potential!

[http://www.cse.secs.oakland.edu/haskell/Special Relativity and Maxwells Equations.pdf](http://www.cse.secs.oakland.edu/haskell/Special%20Relativity%20and%20Maxwells%20Equations.pdf)

- Also...

## Kaluza–Klein theory

From Wikipedia, the free encyclopedia

*This article is about gravitation and electromagnetism. For the mathematical generalization of *K theory*, see *KK-theory*.*

In physics, **Kaluza–Klein theory** (**KK theory**) is a model that seeks to unify the two fundamental forces of gravitation and electromagnetism. The theory was first published in 1921. It was proposed by the mathematician Theodor Kaluza who extended general relativity to a five-dimensional spacetime. The resulting equations can be separated into further sets of equations, one of which is equivalent to Einstein field equations, another set equivalent to Maxwell's equations for the electromagnetic field and the final part an extra scalar field now termed the "radion".

[http://en.wikipedia.org/wiki/Kaluza-Klein\\_theory](http://en.wikipedia.org/wiki/Kaluza-Klein_theory)

# Ok let's do some deriving

First approximation (pretty much exact)

$$\nabla \cdot D = 0$$

$$D = \epsilon_0 E + P$$

$$\nabla \times H = \partial_t (\epsilon_0 E + P)$$

$$\nabla \cdot B = 0$$

$$B = \mu_0 H$$

$$\nabla \times H = \partial_t D$$

$$\nabla \times E = -\mu_0 \partial_t H$$

$$\nabla \times E = -\partial_t B$$

$$\nabla \times \nabla \times E = -\mu_0 \partial_t (\nabla \times H)$$

Bam! Wave Equation

$$\nabla \times \nabla \times E = -\frac{1}{c^2} \partial_t^2 E - \mu_0 \partial_t^2 P$$



No known analytical solutions (by a long shot)....

$$\nabla \times \nabla \times E = -\frac{1}{c^2} \partial_t^2 E - \mu_0 \partial_t^2 P$$

Brace Yourselves

Approximations Are Coming



No known analytical solutions (by a long shot)....

$$\nabla \times \nabla \times E = -\frac{1}{c^2} \partial_t^2 E - \mu_0 \partial_t^2 P$$

$$\nabla \times \nabla \times E = -\nabla^2 E + \nabla(\nabla \cdot E) \leftarrow \text{Use this vector identity}$$


$$-\nabla^2 E + \nabla(\nabla \cdot E) = -\frac{1}{c^2} \partial_t^2 E - \mu_0 \partial_t^2 P$$

$$-\nabla^2 E = -\frac{1}{c^2} \partial_t^2 E - \mu_0 \partial_t^2 P \quad \leftarrow \text{Throw away this term (Approximation!)}$$

**Things are starting to look manageable.. But wait! There's more**

$$-\nabla^2 E = -\frac{1}{c^2} \partial_t^2 E - \mu_0 \partial_t^2 P$$

$$P = P_L + P_{NL}$$



Linear                  Nonlinear

Where we have the following relations:

$$P_L(r, t) = \epsilon_0 \int_{-\infty}^t \chi^{(1)}(t - t') \cdot E(r, t') dt'$$

$$P_{NL}(r, t) = \epsilon_0 \int_{-\infty}^t dt_1 \int_{-\infty}^t dt_2 \int_{-\infty}^t dt_3 \chi^{(3)}(t - t_1', t - t_2', t - t_3') \cdot E(r, t_1) \cdot E(r, t_2) \cdot E(r, t_3)$$

$$\nabla^2 E - \frac{1}{\epsilon_0 c^2} \partial_t^2 D = \mu_0 \partial_t^2 P_{NL}$$

using  $D = \epsilon_0 E + P$

**Ok... Still no analytic solutions known....**

# Who needs extra dimensions?

Unfortunately we can't just throw them away for no reason...

Let's look for a bit at linear propagation to get some ideas of when  
It's valid to throw them away

$$\nabla^2 E - \frac{1}{\epsilon_0 c^2} \partial_t^2 D = 0$$

Makes a lot of sense to look at this equation in the frequency domain

$$\tilde{E}(r, \omega) = \int_{-\infty}^{\infty} E(r, t) \exp(i\omega t) dt$$

Define a Fourier Transform

using

$$\tilde{D}^{(1)} = \epsilon_0 \epsilon^{(1)} \tilde{E}$$

$$\nabla^2 \tilde{E} + \epsilon^{(1)}(\omega) \frac{\omega^2}{c^2} \tilde{E} = 0$$

# Quick Synopsis of what just happened

Multiply by  $\exp(i\omega t)$  and then integrate over all  $t$

$$\nabla^2 E - \frac{1}{\epsilon_0 c^2} \partial_t^2 D = 0$$

$$\int \nabla^2 E e^{i\omega t} dt - \frac{1}{\epsilon_0 c^2} \int \partial_t^2 D e^{i\omega t} dt = 0$$

Switch derivatives with integrals

and use

$$\tilde{E}(r, \omega) = \int_{-\infty}^{\infty} E(r, t) \exp(i\omega t) dt$$

$$\nabla^2 \tilde{E} - \frac{1}{\epsilon_0 c^2} \int (\partial_t^2 D) e^{i\omega t} dt = 0$$

The second term is not so nice

$$(\partial_t^2 D) e^{i\omega t} = \partial_t^2 (D e^{i\omega t}) + \omega^2 e^{i\omega t} D - 2i\omega e^{i\omega t} \partial_t D$$

Do this again, and eventually...

$$\tilde{D}(r, \omega) = \int_{-\infty}^{\infty} D(r, t) \exp(i\omega t) dt$$

$$\nabla^2 \tilde{E} + \epsilon^{(1)}(\omega) \frac{\omega^2}{c^2} \tilde{E} = 0$$

with

$$\epsilon(\omega) = 1 + \tilde{\chi}(\omega)$$

$x$ . However, Lebesgue integration theory has a very powerful criterion called *Lebesgue Dominated Convergence Theorem (LDCT)*. This tells us that if the limit in Eq. 5 exists for *almost all*  $x$ , and there is a function  $H(x) \geq 0$ ,  $\int_a^b H(x) dx < \infty$ , such that

$$\left| \frac{f(x, t + \tau) - f(x, t)}{\tau} \right| \leq H(x), \quad (6)$$

then Eq. 1 holds.

$$\epsilon(\omega) = 1 + \tilde{\chi}(\omega) = (n + i\alpha c / 2\omega)^2 \quad \text{But } \alpha \approx 0 \rightarrow \epsilon(\omega) \approx n^2(\omega)$$

$$\nabla^2 \tilde{E} + n^2(\omega) \frac{\omega^2}{c^2} \tilde{E} = 0 \quad \text{Helmholtz Eqn.}$$

The next part of this story is gunna go down exactly the way you think

Because we're talking about fibers, let's write this in cylindrical coordinates

$$\frac{\partial^2 \tilde{E}}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \tilde{E}}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 \tilde{E}}{\partial \phi^2} + \frac{\partial^2 \tilde{E}}{\partial z^2} + n^2(\omega) \frac{\omega^2}{c^2} \tilde{E} = 0$$

Solve by separation of variables for one component of the field:

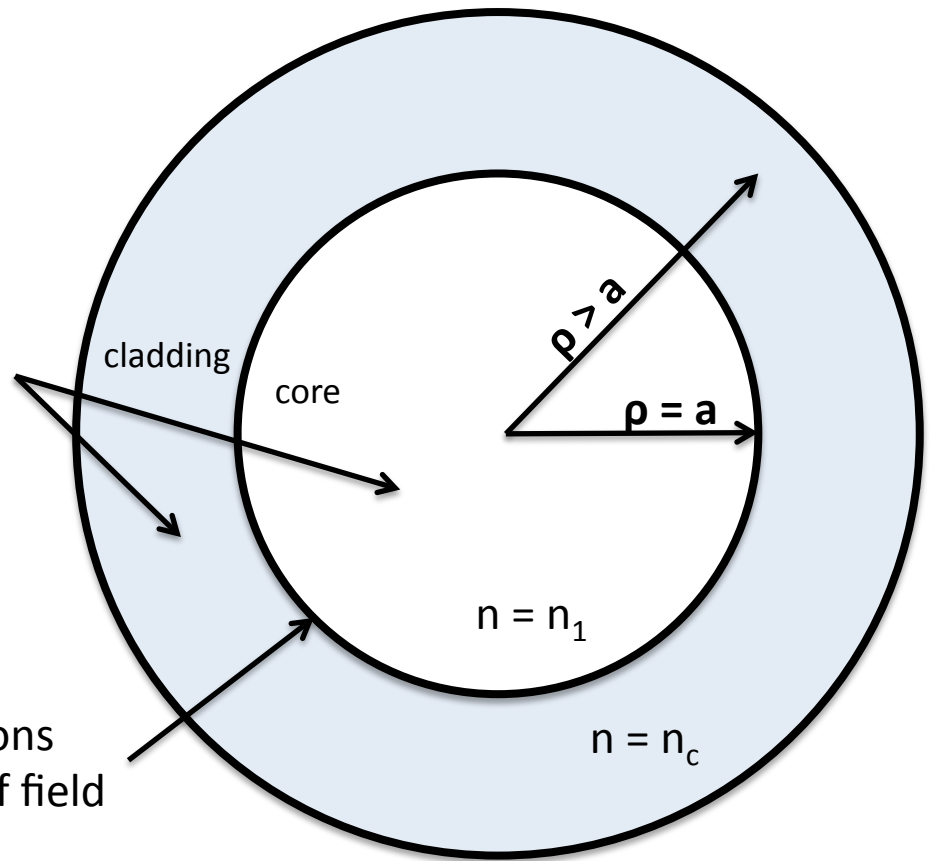
$$\tilde{E}_z(r, \omega) = A(\omega) F(\rho) \exp(im\phi) \exp(i\beta z)$$

<http://mathworld.wolfram.com/Laplacian.html>

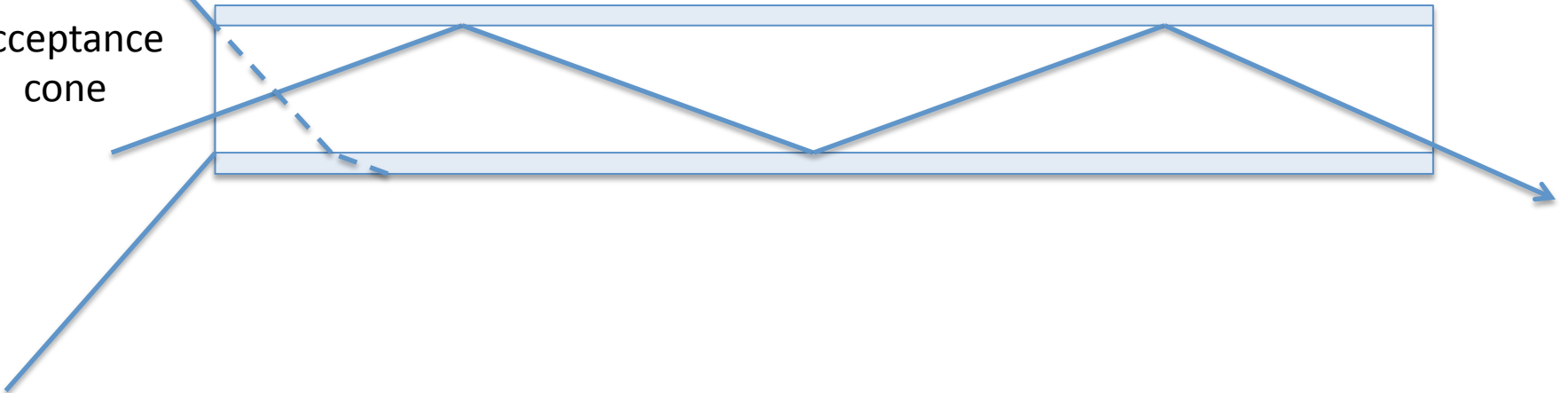
# Picture the Problem

Solve for function in two regions

Match boundary conditions  
on tangential component of field



acceptance  
cone





$$\tilde{E}_z(r, \omega) = A(\omega)F(\rho)\exp(im\phi)\exp(i\beta z)$$

Radial Function

Solves this equation

$$\partial_\rho^2 F(\rho) + \frac{1}{\rho} \partial_\rho F(\rho) + \left( \frac{n^2 \omega^2}{c^2} - \frac{m^2}{\rho^2} - \beta^2 \right) F(\rho) = 0$$

Once again... Function defined by the differential equation it solves

$$c_1 J_m(-i\rho n) + c_2 N_m(-i\rho n) \quad n = \sqrt{\beta^2 - \frac{n^2 \omega^2}{c^2}}$$

Want propagating solution in the core  $\rho \leq a$

$$c_1 J_m(p\rho) + c_2 N_m(p\rho) \quad p = \sqrt{\frac{n_1^2 \omega^2}{c^2} - \beta^2}$$

$c_2$  is zero because we want something that models reality

Want exp. decay in the cladding  $\rho > a$

$$c_3 I_m(q\rho) + c_4 K_m(q\rho) \quad q = \sqrt{\beta^2 - \frac{n_c^2 \omega^2}{c^2}}$$

Modified Bessel Functions (1<sup>st</sup> and 2<sup>nd</sup> kind),  $c_3$  is zero because  $I_m$  is growing

Now we match the boundary conditions for tangential portions of  $\tilde{E}$  &  $\tilde{H}$

i.e.  $\tilde{E}_z, \tilde{H}_z, \tilde{E}_\phi$ , and  $\tilde{H}_\phi$  all must match at  $\rho = a$ , Eigenvalue equation of the form

$$\left[ \frac{J'_m(pa)}{pJ_m(pa)} + \frac{K'_m(pa)}{qK_m(pa)} \right] \left[ \frac{J'_m(pa)}{pJ_m(pa)} + \frac{n_c^2}{n_1^2} \frac{K'_m(pa)}{qK_m(pa)} \right] = \left( \frac{m\beta k_0(n_1^2 - n_c^2)}{an_1 p^2 q^2} \right)^2$$

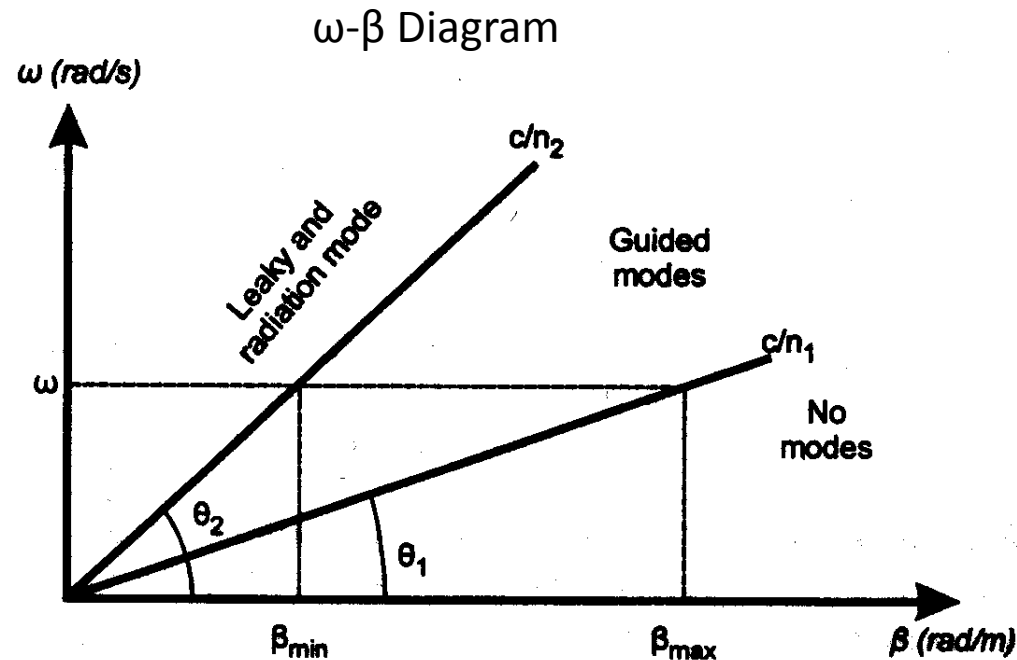
We also need 
$$p^2 + q^2 = (n_1^2 - n_c^2)k_0^2$$

There are several solutions for each  $\beta$  which we write as  $\beta_{lm}$  ( $l, m$  are integers)

Also, define a normalized frequency 
$$V = p_c a = k_0 a (n_1^2 - n_c^2)^{1/2}$$

$$V = p_c a = k_0 a (n_1^2 - n_c^2)^{1/2}$$

Used to determine how many different modes a fiber can support



**Straight lines of  $d\omega/d\beta$  correspond to the group velocity of the different modes. The group velocities of the guided modes all lie between the phase velocities for plane waves in the core or cladding  $c/n_1$  and  $c/n_c$**

# Fiber Modes

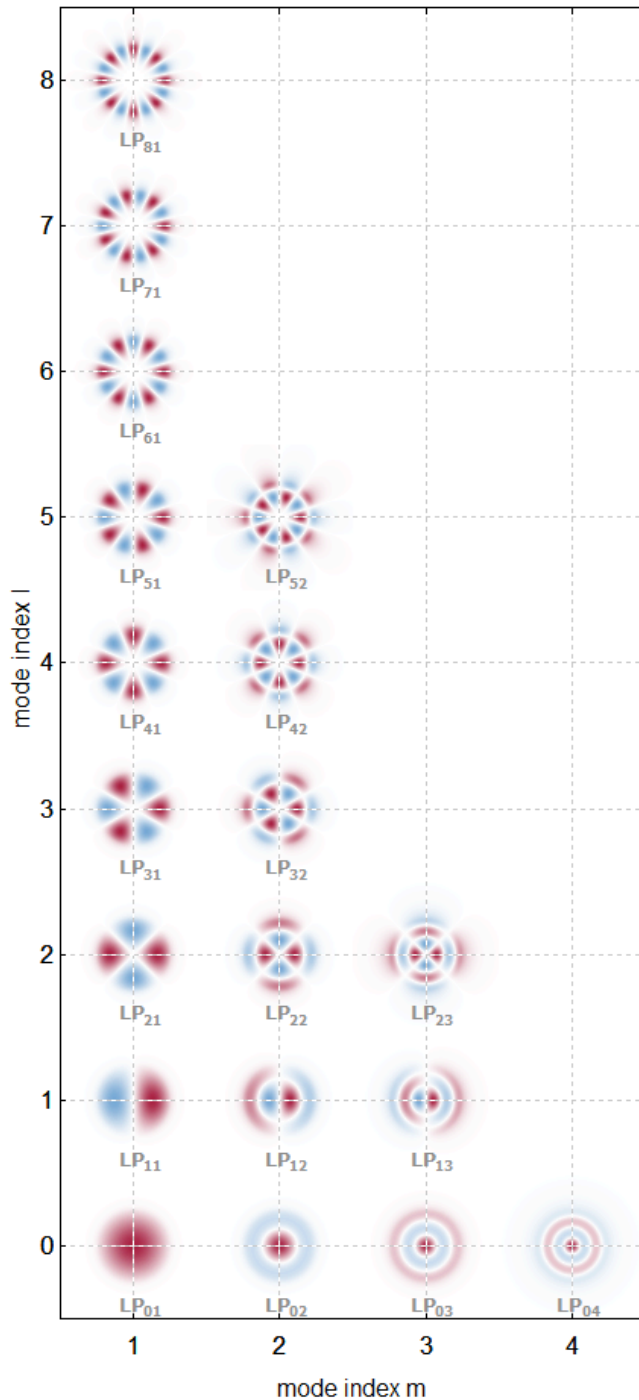
If either  $\tilde{E}_z, \tilde{H}_z$  are zero then we recover

TE<sub>lm</sub> or TM<sub>lm</sub> modes

If neither are zero then we get hybrid modes

$$V = k_0 a NA \quad \text{NA = numerical aperture}$$

$V < 2.5$  single mode. Number of modes scales with  $V^2$



<http://www.rp-photonics.com/fibers.html>

# Mode Properties

- The lowest order mode has 3-components but one of them dominates
- Approximately linearly polarized
- Fundamental Mode  $\sim J_0(p \rho)$  ( $\rho \leq a$ )
- Generally approximated as a Gaussian
- Modes propagate independently in multimode configuration
- Nonlinearity couples them just like independent frequencies

# Nonlinear Wave Equation (again...)

$$\nabla^2 E - \frac{1}{\epsilon_0 c^2} \partial_t^2 D = \mu_0 \partial_t^2 P_{NL}$$

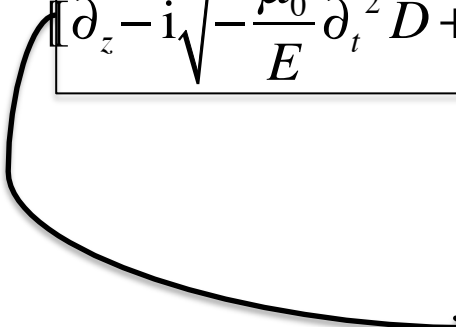
Separate parallel and perpendicular derivatives

$$\llbracket \partial_z^2 + \nabla_\perp^2 - \frac{\mu_0}{E} \partial_t^2 D - \frac{\mu_0}{E} \partial_t^2 P_{NL} \rrbracket E = 0$$

d'Alembert's solution!

$$\boxed{\llbracket \partial_z - i \sqrt{-\frac{\mu_0}{E} \partial_t^2 D + \nabla_\perp^2 - \frac{\mu_0}{E} \partial_t^2 P_{NL}} \rrbracket} \llbracket \partial_z + i \sqrt{-\frac{\mu_0}{E} \partial_t^2 D + \nabla_\perp^2 - \frac{\mu_0}{E} \partial_t^2 P_{NL}} \rrbracket E = 0$$

Investigate forward traveling waves



$$\partial_z E - i \sqrt{-\frac{\mu_0}{E} \partial_t^2 D + \nabla_\perp^2 - \frac{\mu_0}{E} \partial_t^2 P_{NL}} E = 0$$

$$\partial_z E - i \sqrt{-\frac{\mu_0}{E} \partial_t^2 D + \nabla_\perp^2 - \frac{\mu_0}{E} \partial_t^2 P_{NL}} E = 0$$

Immediately expand in a Taylor series

$$i \partial_z E \approx -HE - \frac{1}{2} H^{-1} \left( \nabla_\perp^2 - \frac{\mu_0}{E} \partial_t^2 P_{NL} \right) E \quad \text{with} \quad H^2 \equiv -\frac{\mu_0}{E} \partial_t^2 D$$

Insert Envelope forms into the equation

$$E = E_0(\mathbf{r}, t) e^{i(\beta_0 z - \omega_0 t)} \quad D = D_0(\mathbf{r}, t) e^{i(\beta_0 z - \omega_0 t)} \quad P_{NL} = P_0(\mathbf{r}, t) e^{i(\beta_0 z - \omega_0 t)}$$

$$i \partial_z E_0 = \beta_0 E_0 - H_0 E_0 - \frac{1}{2} H_0^{-1} \left[ \nabla_\perp^2 E_0 + \mu_0 \omega_0^2 \hat{T}^2 P_0 \right] \quad \text{with} \quad \hat{T} \equiv \left( 1 + \frac{i}{\omega_0} \partial_t \right)$$

At this point we shift to a retarded frame  
that travels at a group velocity of  $1/\beta_1$

$$\tau = t - \beta_1 z$$

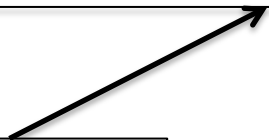
$$i\partial_z E_0 = (\beta_0 + i\beta_1 \partial_t) E_0 - H_0 E_0 - \frac{1}{2} H_0^{-1} \left[ \nabla_{\perp}^2 E_0 + \mu_0 \omega_0^2 \hat{T}^2 P_0 \right]$$

Now Fourier transform to the frequency/spatial-frequency domain  
using the same technique as before


Things simplify a lot and we'll be able to discuss some terms...

$$i\partial_z \tilde{E}_0 = - \boxed{\left[ \beta(\omega) - (\beta_0 + \beta_1 \Delta\omega) \right]} \tilde{E}_0 + \boxed{\left[ 2\pi^2 \frac{c}{n\omega} (f_x^2 + f_y^2) \right]} \tilde{E}_0 - \boxed{\frac{c\mu_0\omega}{2n} \tilde{P}_0}$$


Dispersion



Diffraction



Nonlinearity



Let's talk about these things individually



# But First!

Fourier transform back to the time domain and  
non-dimensionalize this thing  
(This time we'll use a trick)

$$i\partial_z \tilde{E}_0 = -[\beta(\omega) - (\beta_0 + \beta_1 \Delta\omega)] \tilde{E}_0 + \left[ 2\pi^2 \frac{c}{n\omega} (f_x^2 + f_y^2) \right] - \frac{c\mu_0\omega}{2n} \tilde{P}_0$$

$$i\partial_t \leftrightarrow \Delta\omega \quad \text{and} \quad -4\pi^2 (f_x^2 + f_y^2) \leftrightarrow \nabla_\perp^2$$

Leads to

$$i\partial_z E_0 = \frac{1}{2} \beta_2(\omega_0) \partial_t^2 \tilde{E}_0 - \frac{c}{2n\omega_0} \nabla_\perp^2 \tilde{E}_0 - \frac{3}{2} \frac{\omega_0}{nc} \chi^{(3)} |E_0|^2 E_0$$

Note that  $\Delta\omega$  translates to time derivatives, to simplify things we've Taylor expanded

$$\beta(\omega) \approx \beta(\omega_0) + \beta_1(\omega_0)\Delta\omega + \frac{1}{2}\beta_2(\omega_0)\Delta\omega^2 + \dots +$$

$$\beta_k = \frac{\partial^k \beta(\omega)}{\partial \omega^k} \Big|_{\omega=\omega_0}$$


Agrawal suggest normalizing the E-field envelope by  $\sqrt{P_0/A_{eff}}$

That is:  $A_0 = \frac{E_0}{\sqrt{P_0/A_{eff}}}$  and inserting into  $i\partial_z E_0 = \frac{1}{2}\beta_2(\omega_0)\partial_t^2 \tilde{E}_0 - \frac{c}{2n\omega_0}\nabla_\perp^2 \tilde{E}_0 - \frac{3}{2}\frac{\omega_0}{nc}\chi^{(3)}|E_0|^2 E_0$


$$i\partial_z A_0 \sqrt{P_0/A_{eff}} = \frac{1}{2}\beta_2(\omega_0)\partial_t^2 A_0 \sqrt{P_0/A_{eff}} - \frac{c}{2n\omega_0}\nabla_\perp^2 A_0 \sqrt{P_0/A_{eff}} - \frac{3}{2}\frac{\omega_0}{nc}\chi^{(3)}|A_0 \sqrt{P_0/A_{eff}}|^2 A_0 \sqrt{P_0/A_{eff}}$$

Cancelling


$$i\partial_z A_0 = \boxed{\frac{1}{2}\beta_2(\omega_0)\partial_t^2 A_0} - \boxed{\frac{c}{2n\omega_0}\nabla_\perp^2 A_0} - \boxed{\frac{3}{2}\frac{\omega_0}{nc}\chi^{(3)}\frac{P_0}{A_{eff}}|A_0|^2 A_0}$$



Dispersion



Diffraction



Nonlinearity

where the effective mode are is defined as

$$A_{eff} = \frac{(\iint |F(x,y)|^2 dx dy)^2}{\iint |F(x,y)|^4 dx dy}$$

Now we can define some non-dimensional parameters

Define a non-dimensional time by introducing the normalization:  $t = \hat{t} \tau$

Where  $\tau$  is the characteristic pulse width

Derivatives in time are changed to read  $\frac{\partial}{\partial t} = \left( \frac{\partial \hat{t}}{\partial t} \right) \frac{\partial}{\partial \hat{t}} = \frac{1}{\tau} \frac{\partial}{\partial \hat{t}}$

Inserting into our equation we get for the dispersion term  $\frac{1}{2} \beta_2 \partial_t^2 A_0 \rightarrow \frac{1}{2} \frac{\beta_2}{\tau^2} \partial_{\hat{t}}^2 A_0$

$$i \partial_z A_0 = \boxed{\frac{1}{2} \frac{\beta_2}{\tau^2} \partial_{\hat{t}}^2 A_0} - \frac{c}{2n\omega_0} \nabla_{\perp}^2 A_0 - \frac{4n_2\omega_0}{cA_{eff}} P_0 |A_0|^2 A_0$$

Introduce the Dispersion Length  $L_D \equiv \frac{\tau^2}{|\beta_2|}$  and inserting

$$i \partial_z A_0 = \frac{1}{2} \frac{\text{sgn}(\beta_2)}{L_D} \partial_{\hat{t}}^2 A_0 - \frac{c}{2n_0\omega_0} \nabla_{\perp}^2 A_0 - \frac{4n_2\omega_0}{cA_{eff}} P_0 |A_0|^2 A_0$$

Introduce the Diffraction Length  $L_{Dif} \equiv z_R = \frac{\pi w_0^2}{\lambda} = \frac{1}{2} \beta_0 w_0^2$  and inserting

Derivatives on z are changed to:  $\frac{\partial}{\partial z} = \left( \frac{\partial \hat{z}}{\partial z} \right) \frac{\partial}{\partial \hat{z}} = \frac{1}{L_{Dif}} \frac{\partial}{\partial \hat{z}}$

$$i \frac{1}{L_{Dif}} \partial_{\hat{z}} A_0 = \frac{1}{2} \frac{\text{sgn}(\beta_2)}{L_D} \partial_{\hat{t}}^2 A_0 - \frac{c}{2n_0 \omega_0} \nabla_{\perp}^2 A_0 - \frac{4n_2 \omega_0}{c A_{eff}} P_0 |A_0|^2 A_0$$

Similarly define a length scale for the transverse directions  $\hat{x} = \frac{2x}{w_0}$  and inserting

$$\frac{\partial}{\partial x} = \left( \frac{\partial \hat{x}}{\partial x} \right) \frac{\partial}{\partial \hat{x}} = \frac{2}{w_0} \frac{\partial}{\partial \hat{x}}$$

$$i \frac{1}{L_{Dif}} \partial_z A_0 = \frac{1}{2} \frac{\text{sgn}(\beta_2)}{L_D} \partial_t^2 A_0 - \frac{1}{L_{Dif}} \hat{\nabla}_{\perp}^2 A_0 - 4 \frac{n_2 \omega_0}{c A_{eff}} P_0 |A_0|^2 A_0$$

Finally define a nonlinear length scale  $L_{NL} = \frac{1}{\gamma P_0}$  where  $\gamma \equiv 4 \frac{n_2 \omega_0}{c A_{eff}} [=] \text{meters}$  and inserting

$$i \frac{1}{L_{Dif}} \partial_{\hat{z}} A_0 = \frac{1}{2} \frac{\text{sgn}(\beta_2)}{L_D} \partial_{\hat{t}}^2 A_0 - \frac{1}{L_{Dif}} \hat{\nabla}_{\perp}^2 A_0 - \frac{1}{L_{NL}} |A_0|^2 A_0$$

# So now we can talk about Physics!

$$i \frac{1}{L_{Dif}} \partial_z A_0 = \frac{1}{2} \frac{\text{sgn}(\beta_2)}{L_D} \partial_t^2 A_0 - \frac{1}{L_{Dif}} \hat{\nabla}_\perp^2 A_0 - \frac{1}{L_{NL}} |A_0|^2 A_0$$

Take a quick look when there's only diffraction and the nonlinearity

$$i \partial_z A_0 = -\hat{\nabla}_\perp^2 A_0 - \frac{L_{Dif}}{L_{NL}} |A_0|^2 A_0$$

The nonlinear Schrödinger equation is a **partial differential equation**, applicable to **classical** and **quantum mechanics**.

## Classical equation

The classical field equation (in **dimensionless** form) is:<sup>[1]</sup>

**Nonlinear Schrödinger equation** (*Classical mechanics*)

$$i \partial_t \psi = -\frac{1}{2} \partial_x^2 \psi + \kappa |\psi|^2 \psi$$

[http://en.wikipedia.org/wiki/Nonlinear\\_Schrodinger\\_equation](http://en.wikipedia.org/wiki/Nonlinear_Schrodinger_equation)

# A note about length scales

$$L_D = \frac{\tau^2}{|\beta_2|}, \quad L_{Dif} \equiv \frac{1}{2} \beta_0 w_0^2, \quad L_{NL} = \frac{1}{\gamma P_0} \text{ where } \gamma \equiv 4 \frac{n_2 \omega_0}{c A_{eff}}$$

The diagram shows three mathematical expressions for length scales. Below each expression is a label in a rectangular box: 'Dispersion Length' under  $L_D$ , 'Diffraction Length' under  $L_{Dif}$ , and 'Nonlinear Length' under  $L_{NL}$ . Arrows point from each label box to its corresponding formula.

The shortest length tends to dominate the propagation

Neglect terms to see what other terms do

$$i \frac{1}{L_{Dif}} \partial_z A_0 = \frac{1}{2} \frac{\text{sgn}(\beta_2)}{L_D} \partial_t^2 A_0 - \frac{1}{L_{Dif}} \hat{\nabla}_\perp^2 A_0 - \frac{1}{L_{NL}} |A_0|^2 A_0$$

First let's take a look at just the nonlinear term

$$i \partial_z A_0 = - \boxed{\frac{L_{Dif}}{L_{NL}}} |A_0|^2 A_0 \quad \text{Scaling goes like a ratio of the lengths}$$

Revert to a dimension-full version of this term

$$i \partial_z E_0 = - \frac{3\omega_0}{2cn_0} \chi^{(3)} |E_0|^2 E_0 = -2\beta_0 \frac{n_2}{n_0} I_0 E_0$$

This equation has a formal solution of the form

$$E_0(r, dz) = \exp(-2dz\beta_0 \frac{n_2}{n_0} I_0) E_0(r, 0)$$

With nonlinear phase accumulation

$$\boxed{\phi_{NL} = 2\beta_0 \frac{n_2}{n_0} I_0 dz} \quad \text{B-Integral!}$$

# Self-Focusing

$$\phi_{NL} = 2\beta_0 \frac{n_2}{n_0} I_0 dz$$

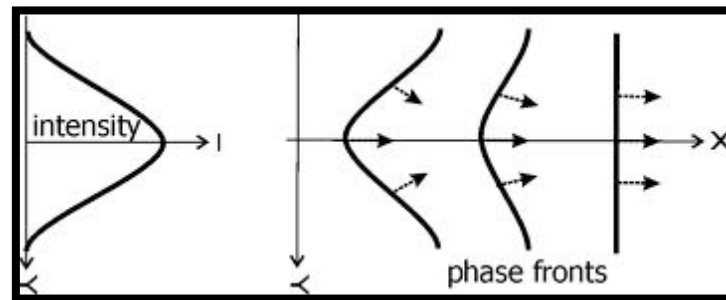
Consider the case when  $I_0$  has the shape of a Gaussian

$$I_0 = \frac{a_0}{2} \exp(-r^2 / r_0^2) \longrightarrow \phi_{NL} = a_0 \beta_0 \frac{n_2}{n_0} \exp(-r^2 / r_0^2) dz$$

Taylor expand the Gaussian profile

$$\phi_{NL} = -a_0 \beta_0 \frac{n_2}{n_0} \frac{r^2}{r_0^2} dz = -\beta_0 \frac{r^2}{2f} \quad f = \frac{2r_0^2}{dz} \frac{n_0}{a_0 n_2}$$

Quadratic phase associated with focusing





# Self-Phase Modulation

$$\phi_{NL} = 2\beta_0 \frac{n_2}{n_0} I_0 dz$$

Consider the case when  $I_0$  has the shape of a Gaussian

$$I_0 = \frac{a_0}{2} \exp(-t^2 / \tau^2) \longrightarrow \phi_{NL} = a_0 \beta_0 \frac{n_2}{n_0} \exp(-t^2 / \tau^2) dz$$

Let's take a look at the instantaneous color as a function of time in the pulse

$$\omega_{inst} = \omega_0 - \partial_t \phi_{NL} = \omega_0 - a_0 \beta_0 dz \frac{n_2}{n_0} \frac{2t}{\tau^2} \exp(-t^2 / \tau^2)$$

Again... Taylor expand the Gaussian profile

$$\omega_{inst} \approx \omega_0 + bt \qquad b = 2a_0 \beta_0 dz \frac{n_2}{n_0 \tau^2}$$

# Self-Phase Modulation

$$\omega_{inst} = \omega_0 \left[ 1 - 2a_0 dz \frac{n_2}{c \tau^2} t \right]$$

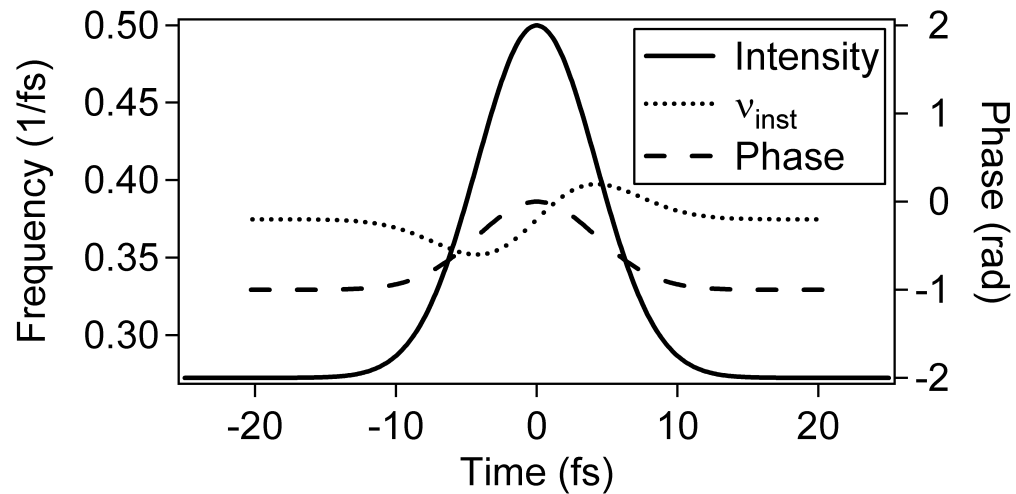
Effect is similar to the Doppler shift for an object moving at speed  $v_z$

$$z + v_z t \quad \phi = \beta_0 (z + v_z t) - \omega_0 t$$

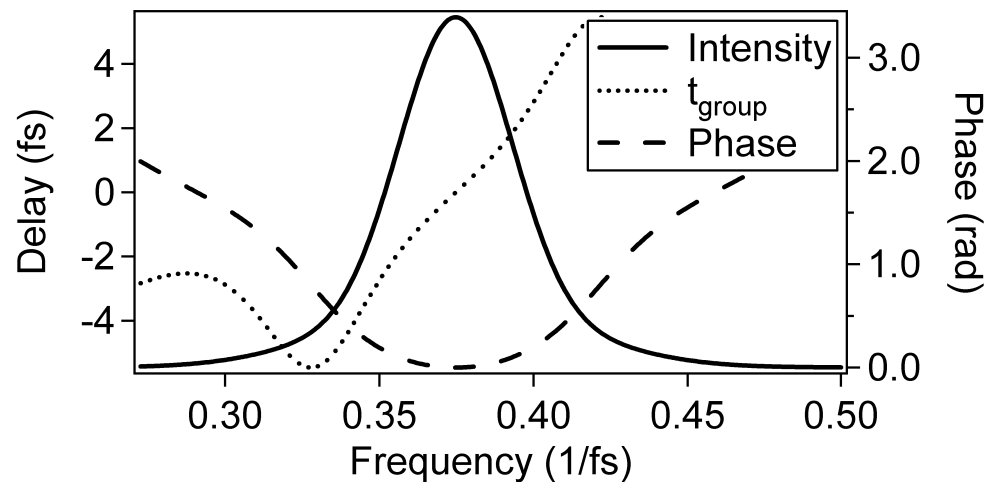
$$\Delta\omega = -\frac{\partial\phi}{\partial t} = -\beta_0 v_z + \omega_0 = \omega_0 (1 - v_z / c)$$

$$\omega_{inst} = \omega_0 (1 - v / c) \rightarrow v = 2 \frac{dz}{\tau} a_0 n_2$$

# Self-Phase Modulation



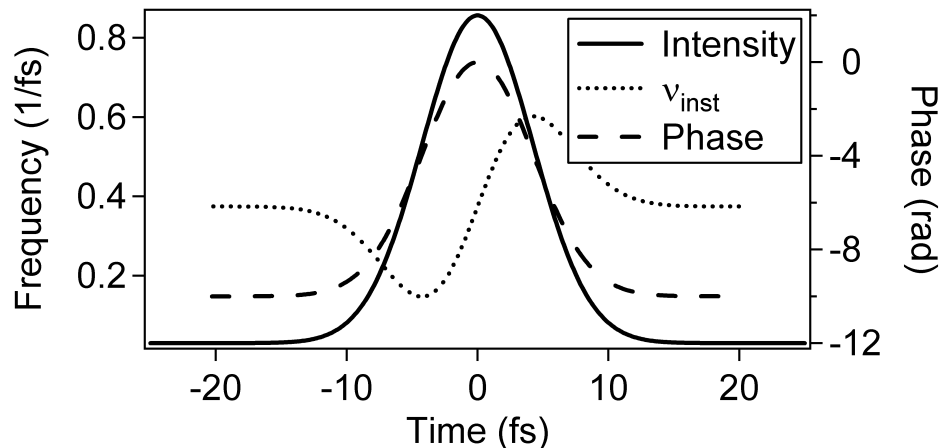
Time domain pulse



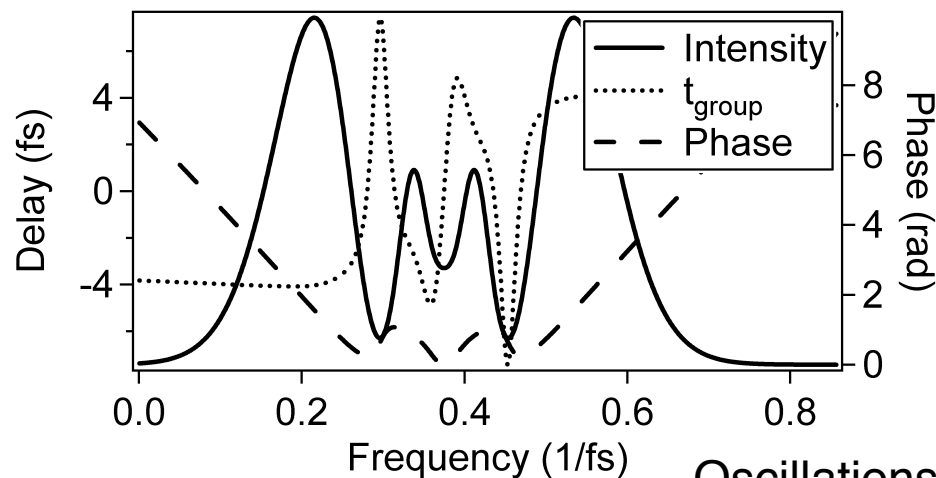
Frequency domain pulse

Adapted from Rick Trebino

# (a lot of) Self-Phase Modulation



Time domain pulse



Frequency domain pulse

Oscillations occur in spectrum because all frequencies occur twice and interfere, except for inflection points, which yield maximum and minimum frequencies.

# Operator Form NLS

$$i \frac{1}{L_{Dif}} \partial_z A_0 = \frac{1}{2} \frac{\text{sgn}(\beta_2)}{L_D} \partial_t^2 A_0 - \frac{1}{L_{Dif}} \hat{\nabla}_\perp^2 A_0 - \frac{1}{L_{NL}} |A_0|^2 A_0$$

Assume single mode propagation in a fiber

$$i \frac{1}{L_{Dif}} \partial_z A_0 = \frac{1}{2} \frac{\text{sgn}(\beta_2)}{L_D} \partial_t^2 A_0 - \frac{1}{L_{NL}} |A_0|^2 A_0$$

Or

$$\partial_z A_0 = -i \frac{\text{sgn}(\beta_2)}{2} \frac{L_{Dif}}{L_D} \partial_t^2 A_0 + i \frac{L_{Dif}}{L_{NL}} |A_0|^2 A_0$$

Define Operators

$$\hat{D} \equiv -i \frac{\text{sgn}(\beta_2)}{2} \frac{L_{Dif}}{L_D} \partial_t^2$$

$$\hat{N} \equiv i \frac{L_{Dif}}{L_{NL}} |A_0|^2$$

Dispersion

$$\partial_z A_0 = [\hat{D} + \hat{N}] A_0$$

Nonlinearity

# Split Step Fourier Method

- We assume the distance  $h$  we are propagating is very small
- This allows us to say that the dispersion and nonlinearity act separately
- Propagate in two steps
  - First step, only nonlinearity (  $\hat{D} = 0$  )
  - Second step, only dispersion only (  $\hat{N} = 0$  )

This allows us to write  $A_0(z+h,t) \approx \exp(h\hat{D})\exp(h\hat{N})A(z,t)$

Unfortunately exponentials of operators can be tricky... but don't worry

$$\exp(h\hat{D})B(z,t) = \mathcal{F}^{-1} \{ \exp[h\hat{D}(-i\Delta\omega)] \mathcal{F} \{ B(z,t) \} \}$$

This is helpful because...

$$\hat{D}(-i\Delta\omega) = \text{Exp} \left[ -\frac{\text{sgn}(\beta_2)}{2} \frac{L_{\text{Dif}}}{L_D} \Delta\omega \right]$$

...is just a number now

Define Soliton Number

$$N_s = \sqrt{L_D / L_{NL}} = \sqrt{\gamma P_0 \tau^2 / |\beta_2|}$$

# What's a Soliton?

## Definition

<http://en.wikipedia.org/wiki/Soliton#Definition> [\[edit\]](#)

A single, consensus definition of a soliton is difficult to find. [Drazin & Johnson \(1989, p. 15\)](#) ascribe three properties to solitons:

1. They are of permanent form;
2. They are localised within a region;
3. They can interact with other solitons, and emerge from the collision unchanged, except for a [phase shift](#).

More formal definitions exist, but they require substantial mathematics. Moreover, some scientists use the term *soliton* for phenomena that do not quite have these three properties (for instance, the 'light bullets' of [nonlinear optics](#) are often called solitons despite losing energy during interaction).

$$i \partial_z A_0 = \frac{1}{2} \beta_2 \partial_t^2 A_0 - \gamma_1 |A_0|^2 A_0 \quad \gamma_1 \propto n_2$$

$$\tilde{A}_s(z, t) = A_s^0 \operatorname{sech}(t / \tau) \exp(i\kappa z)$$

We can show that this is a solution by plugging it in

$$\kappa = -\frac{\beta_2}{2\tau^2} = \frac{1}{2} \gamma_1 |A_s^0|^2$$

Note here that the group delay dispersion and nonlinearity have opposite signs



The NLS can model propagation pretty well

