# EM waves: energy, resonators

Scalar wave equation

Maxwell equations to the EM wave equation

A simple linear resonator

Energy in EM waves

3D waves

# Simple scalar wave equation

- $2^{nd}$ order PDE $\frac{\partial^2}{\partial z^2}\psi(z,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(z,t) = 0$

- Assume separable solution $\psi(z,t) = f\left(z\right)g\left(t\right)$

$$\frac{1}{f\left(z\right)}\frac{\partial^2}{\partial z^2}f(z) - \frac{1}{c^2}\frac{1}{g\left(t\right)}\frac{\partial^2}{\partial t^2}g\left(t\right) = 0$$

  – Each part is equal to a constant *A*

$$\frac{1}{f\left(z\right)}\frac{\partial^2}{\partial z^2}f(z) = A,\ \frac{1}{c^2}\frac{1}{g\left(t\right)}\frac{\partial^2}{\partial t^2}g\left(t\right) = A$$

$$f(z) = \cos\left(kz\right) \rightarrow -k^2 = A,\ g\left(t\right) = \cos\left(\omega\, t\right) \rightarrow -\omega^2\frac{1}{c^2} = A$$

$$\omega = \pm k\, c$$

Sin( ) also works as a second solution

# Full solution of wave equation

- Full solution is a linear combination of both

$$\psi(z,t) = f\left(z\right)g\left(t\right) = \left(A_1 \cos kz + A_2 \sin kz\right)\left(B_1 \cos \omega t + B_2 \sin \omega t\right)$$

- Too messy: use complex solution instead:

$$\psi(z,t) = f\left(z\right)g\left(t\right) = \left(A_1 e^{ikz} + A_2 e^{-ikz}\right)\left(B_1 e^{i\omega t} + B_2 e^{-i\omega t}\right)$$

$$\psi(z,t) = A_1 B_1 e^{i\left(kz+\omega t\right)} + A_2 B_2 e^{-i\left(kz+\omega t\right)} + A_1 B_2 e^{i\left(kz-\omega t\right)} + A_2 B_1 e^{-i\left(kz-\omega t\right)}$$

  – Constants are arbitrary: rewrite

$$\psi(z,t) = A_1 \cos\left(kz + \omega t + \phi_1\right) + A_2 \cos\left(kz - \omega t + \phi_2\right)$$

# Interpretation of solutions

- Wave vector $k = \dfrac{2\pi}{\lambda}$

- Angular frequency $\omega = 2\pi\nu$

- Wave total phase: $\Phi = kz - \omega t + \phi$
  - "absolute phase": $\phi$
  - Phase velocity: *c* $\Phi = kz - k\,c\,t + \phi = k\left(z - c\,t\right) + \phi$

$$\Phi = \text{constant when } z = ct$$

$$\psi(z,t) = A_1 \cos\left(kz + \omega t + \phi_1\right) + A_2 \cos\left(kz - \omega t + \phi_2\right)$$

Reverse (to -z) Forward (to +z)

# Maxwell's Equations to wave eqn

- The induced polarization, **P**, contains the effect of the medium:

$$\vec{\nabla} \cdot \mathbf{E} = 0 \qquad \vec{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\vec{\nabla} \cdot \mathbf{B} = 0 \qquad \vec{\nabla} \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \mu_0 \frac{\partial \mathbf{P}}{\partial t}$$

Take the curl:

$$\vec{\nabla} \times \left(\vec{\nabla} \times \mathbf{E}\right) = -\frac{\partial}{\partial t}\vec{\nabla} \times \mathbf{B} = -\frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \mu_0 \frac{\partial \mathbf{P}}{\partial t}\right)$$

Use the vector ID:

$$\mathbf{A} \times \left(\mathbf{B} \times \mathbf{C}\right) = \mathbf{B}\left(\mathbf{A} \cdot \mathbf{C}\right) - \mathbf{C}\left(\mathbf{A} \cdot \mathbf{B}\right)$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \mathbf{E}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \mathbf{E}\right) - \left(\vec{\nabla} \cdot \vec{\nabla}\right)\mathbf{E} = -\vec{\nabla}^2 \mathbf{E}$$

$$\boxed{\vec{\nabla}^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2}}$$

"Inhomogeneous Wave Equation"

# Maxwell's Equations in a Medium

- The induced polarization, **P**, contains the effect of the medium:

$$\vec{\nabla}^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = \mu_0\frac{\partial^2\mathbf{P}}{\partial t^2}$$

- Sinusoidal waves of all frequencies are solutions to the wave equation
- The polarization (**P**) can be thought of as the driving term for the solution to this equation, so the polarization determines which frequencies will occur.
- For linear response, **P** will oscillate at the same frequency as the input.

$$\mathbf{P}\left(\mathbf{E}\right) = \varepsilon_0\chi\mathbf{E}$$

- In nonlinear optics, the induced polarization is more complicated:

$$\mathbf{P}\left(\mathbf{E}\right) = \varepsilon_0\left(\chi^{(1)}\mathbf{E} + \chi^{(2)}\mathbf{E}^2 + \chi^{(3)}\mathbf{E}^3 + \ldots\right)$$

- The extra nonlinear terms can lead to new frequencies.

# Solving the wave equation: linear induced polarization

For low irradiances, the polarization is proportional to the incident field:

$$\mathbf{P}(\mathbf{E}) = \varepsilon_0 \chi \mathbf{E}, \quad \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 (1+\chi)\mathbf{E} = \varepsilon \mathbf{E} = n^2 \mathbf{E}$$

In this simple (and most common) case, the wave equation becomes:

$$\vec{\nabla}^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{1}{c^2}\chi \frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad \rightarrow \vec{\nabla}^2 \mathbf{E} - \frac{n^2}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

Using: $\varepsilon_0 \mu_0 = 1/c^2$ $\qquad \varepsilon_0 (1+\chi) = \varepsilon = n^2$

The electric field is a vector function in 3D, so this is actually 3 equations:

$$\vec{\nabla}^2 E_x(\mathbf{r},t) - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2} E_x(\mathbf{r},t) = 0$$

$$\vec{\nabla}^2 E_y(\mathbf{r},t) - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2} E_y(\mathbf{r},t) = 0$$

$$\vec{\nabla}^2 E_z(\mathbf{r},t) - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2} E_z(\mathbf{r},t) = 0$$

# Plane wave solutions for the wave equation

If we assume the solution has no dependence on x or y:

$$\vec{\nabla}^2 \mathbf{E}(z,t) = \frac{\partial^2}{\partial x^2}\mathbf{E}(z,t) + \frac{\partial^2}{\partial y^2}\mathbf{E}(z,t) + \frac{\partial^2}{\partial z^2}\mathbf{E}(z,t) = \frac{\partial^2}{\partial z^2}\mathbf{E}(z,t)$$

$$\rightarrow \frac{\partial^2 \mathbf{E}}{\partial z^2} - \frac{n^2}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

The solutions are oscillating functions, for example

$$\mathbf{E}(z,t) = \hat{\mathbf{x}} E_x \cos(k_z z - \omega t)$$

Where $\omega = k c, \quad k = 2\pi n / \lambda, \quad v_{ph} = c / n$

This is a linearly polarized wave.

# Complex notation for waves

- Write cosine in terms of exponential

$$\mathbf{E}(z,t) = \hat{\mathbf{x}} E_x \cos(kz - \omega t + \phi) = \hat{\mathbf{x}} E_x \frac{1}{2}\left(e^{i(kz-\omega t+\phi)} + e^{-i(kz-\omega t+\phi)}\right)$$

  – Note E-field is a *real* quantity.

  – It is convenient to work with just one part

    - We will use $E_0 e^{+i(kz-\omega t)}$ $\qquad E_0 = \frac{1}{2} E_x e^{i\phi}$
    - Svelto: $e^{-i(kz-\omega t)}$

  – Then take the real part.

    - No factor of 2
    - In *nonlinear* optics, we have to explicitly include conjugate term

# Example: linear resonator (1D)

- Boundary conditions: conducting ends (mirrors)

$$E_x\left(z=0,t\right)=0 \qquad E_x\left(z=L_z,t\right)=0$$

- Field is a superposition of +'ve and –'ve waves:

$$E_x\left(z,t\right)=A_+e^{i\left(k_z z-\omega t+\phi_+\right)}+A_-e^{i\left(-k_z z-\omega t+\phi_-\right)}$$

  – Absorb phase into complex amplitude

$$E_x\left(z,t\right)=\left(A_+e^{+ik_z z}+A_-e^{-ik_z z}\right)e^{-i\omega t}$$

  – Apply b.c. at z = 0

$$E_x\left(0,t\right)=0=\left(A_++A_-\right)e^{-i\omega t}\rightarrow A_+=-A_-$$

$$E_x\left(z,t\right)=A\sin k_z z\; e^{-i\omega t}$$

# Quantization of frequency: longitudinal modes

- Apply b.c. at far end

$$E_x\left(L_z, t\right) = 0 = A\sin k_z L_z\, e^{-i\omega t} \qquad \rightarrow k_z L_z = l\pi \qquad l = 1, 2, 3, \cdots$$

– Relate to wavelength:

$$k_z = \frac{2\pi}{\lambda} = \frac{l\pi}{L_z} \rightarrow L_z = l\frac{\lambda}{2}$$

Integer number of half-wavelengths

– Relate to allowed frequencies:

$$\frac{\omega_l}{c} = \frac{l\pi}{L_z} \rightarrow \nu_l = l\frac{c}{2L_z}$$

– Equally spaced frequencies:

$$\Delta\nu = \frac{c}{2L_z} = \frac{1}{T_{RT}}$$

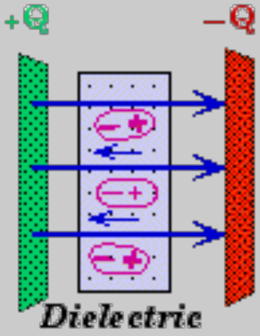
Frequency spacing = 1/ round trip time

# Wave energy and intensity

- Both E and H fields have a corresponding energy density (J/m$^3$)
  - For static fields (e.g. in capacitors) the energy density can be calculated through the work done to set up the field

$$\rho = \tfrac{1}{2}\varepsilon E^2 + \tfrac{1}{2}\mu H^2$$

  - Some work is required to polarize the medium
  - Energy is contained in both fields, but H field can be calculated from E field



# Calculating H from E in a plane wave

- Assume a non-magnetic medium

$$\mathbf{E}(z,t) = \hat{\mathbf{x}} E_x \cos(kz - \omega t)$$

$$\vec{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$$

  – Can see **H** is perpendicular to **E**

$$-\mu_0 \frac{\partial \mathbf{H}}{\partial t} = \vec{\nabla} \times \mathbf{E} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ E_x & 0 & 0 \end{vmatrix} = \hat{\mathbf{y}} \partial_z E_x = -\hat{\mathbf{y}} k_z E_0 \sin(k_z z - \omega t)$$

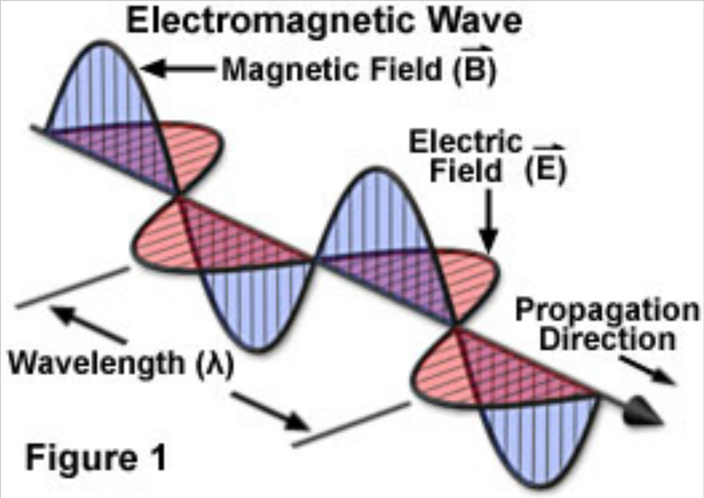
  – Integrate to get **H**-field:

$$\mathbf{H} = \hat{\mathbf{y}} \int \frac{k_z E_0}{\mu_0} \sin(k_z z - \omega t) dt = \hat{\mathbf{y}} \frac{k_z E_0}{\mu_0} \left( \frac{-\cos(k_z z - \omega t)}{-\omega} \right)$$

# H field from E field

- H field for a propagating wave is *in phase* with E-field

$$\mathbf{H} = \hat{\mathbf{y}} H_0 \cos\left(k_z z - \omega t\right)$$
$$= \hat{\mathbf{y}} \frac{k_z}{\omega \mu_0} E_0 \cos\left(k_z z - \omega t\right)$$




- Amplitudes are not independent

$$H_0 = \frac{k_z}{\omega \mu_0} E_0 \qquad k_z = n\frac{\omega}{c} \qquad c^2 = \frac{1}{\mu_0 \varepsilon_0} \rightarrow \frac{1}{\mu_0 c} = \varepsilon_0 c$$

$$H_0 = \frac{n}{c \mu_0} E_0 = n \varepsilon_0 c E_0$$

# Energy density in an EM wave

- Back to energy density, non-magnetic

$$\rho = \tfrac{1}{2}\varepsilon E^2 + \tfrac{1}{2}\mu_0 H^2$$

$$H = n\varepsilon_0 cE$$

$$\varepsilon = \varepsilon_0 n^2$$

$$\rho = \tfrac{1}{2}\varepsilon_0 n^2 E^2 + \tfrac{1}{2}\mu_0 n^2 \varepsilon_0{}^2 c^2 E^2$$

$$\mu_0 \varepsilon_0 c^2 = 1$$

$$\rho = \varepsilon_0 n^2 E^2 = \varepsilon_0 n^2 E^2 \cos^2\left(k_z z - \omega t\right)$$

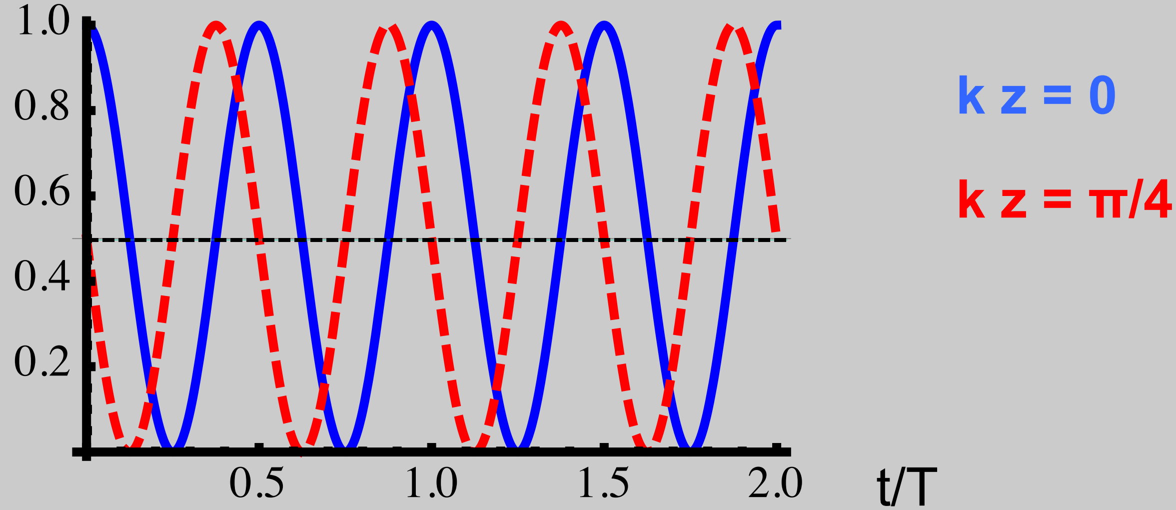
***Equal energy*** in both components of wave

# Cycle-averaged energy density

- Optical oscillations are faster than detectors
- Average over one cycle:

$$\langle \rho \rangle = \varepsilon_0 n^2 E_0{}^2 \frac{1}{T}\int_0^T \cos^2\left(k_z z - \omega t\right) dt$$

– Graphically, we can see this should = ½




– Regardless of position z

$$\langle \rho \rangle = \frac{1}{2}\varepsilon_0 n^2 E_0{}^2$$

# Intensity and the Poynting vector

- Intensity is an energy flux (J/s/cm$^2$)
- In EM the Poynting vector give energy flux

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

  – For our plane wave,

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = E_0 \cos\left(k_z z - \omega t\right) n\varepsilon_0 c E_0 \cos\left(k_z z - \omega t\right) \hat{\mathbf{x}} \times \hat{\mathbf{y}}$$

$$\mathbf{S} = n\varepsilon_0 c E_0^2 \cos^2\left(k_z z - \omega t\right) \hat{\mathbf{z}}$$

  – **S** is along **k**

- Time average: $\mathbf{S} = \frac{1}{2} n \varepsilon_0 c E_0^2 \hat{\mathbf{z}}$
- *Intensity* is the magnitude of **S**

$$I = \frac{1}{2} n\varepsilon_0 c E_0^2 = \frac{c}{n}\rho = V_{phase} \cdot \rho$$

**Photon flux:** $F = \dfrac{I}{h\nu}$

# General plane wave solution

- Assume separable function

$$\mathbf{E}(x,y,z,t) \sim f_1(x) f_2(y) f_3(z) g(t)$$

$$\vec{\nabla}^2 \mathbf{E}(z,t) = \frac{\partial^2}{\partial x^2}\mathbf{E}(z,t) + \frac{\partial^2}{\partial y^2}\mathbf{E}(z,t) + \frac{\partial^2}{\partial z^2}\mathbf{E}(z,t) = \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}(z,t)$$

- Solution takes the form:

$$\mathbf{E}(x,y,z,t) = \mathbf{E_0} e^{ik_x x} e^{ik_y y} e^{ik_z z} e^{-i\omega t} = \mathbf{E_0} e^{i\left(k_x x + k_y y + k_z z\right)} e^{-i\omega t}$$

$$\mathbf{E}(x,y,z,t) = \mathbf{E_0} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

- Now k-vector can point in arbitrary direction