In (1), $\partial\rho/\partial t = 0$. The partial derivative means keeping $x$ fixed. Thus if $x$ is kept fixed, $\rho$ doesn't change with $t$. By integrating, $\rho$ is a constant *for each fixed* $x$. However, for different $x$'s different constants could result. The arbitrary constant now depends on $x$ in an arbitrary way. Hence the constant is an arbitrary function of $x$. In general arbitrary constants of integration become arbitrary functions when the integration is of a partial derivative. Thus

$$\rho = c_0(x),$$

is the general solution of $\partial\rho/\partial t = 0$, where $c_0(x)$ is an arbitrary function of $x$. As a check, $\rho = c_0(x)$ is substituted into the partial differential equation, $\partial\rho/\partial t = 0$, in which case we quickly can verify that $\rho = c_0(x)$ is the solution. To determine the arbitrary function, one initial condition is needed (corresponding to the one initial condition for the ordinary differential equation). The initial condition is the initial value of $\rho(x, t)$, the initial traffic density $\rho(x, 0)$. Can the partial differential equation be solved for any given initial condition, that is for $\rho(x, 0)$ being prescribed, $\rho(x, 0) = f(x)$? Equivalently, can the arbitrary function, $\rho(x, t) = c_0(x)$, be determined such that initially $\rho(x, 0) = f(x)$? In this case it is quite simple as $c_0(x) = f(x)$. Thus

$$\rho(x, t) = f(x)$$

solves the partial differential equation and simultaneously satisfies the initial condition.

We now consider example (2),

$$\frac{\partial\rho}{\partial t} = -\rho + 2e^t.$$

Again we will satisfy the initial condition $\rho(x, 0) = f(x)$. The partial differential equation can again be integrated yielding (for each fixed $x$)

$$\rho = c_1e^{-t} + e^t.$$

As before, the constant can depend on $x$ in an arbitrary way. Hence

$$\rho(x, t) = c_1(x)e^{-t} + e^t.$$

The initial condition is satisfied if $f(x) = c_1(x) + 1$, and hence the solution of problem (2) satisfying the given initial condition is

$$\rho(x, t) = [f(x) - 1]e^{-t} + e^t.$$

For example (3),

$$\frac{\partial\rho}{\partial t} = -x\rho,$$

keeping $x$ fixed (as implied by $\partial/\partial t$) yields the solution of the ordinary differential equation,

$$\rho = c_3e^{-xt}.$$



For other values of $x$ the constant may vary, and hence the solution of the partial differential equation is

$$\rho(x, t) = c_3(x)e^{-xt}.$$

The initial condition, $\rho(x, 0) = f(x)$, is satisfied if $c_3(x) = f(x)$, yielding the solution of the initial value problem,

$$\rho(x, t) = f(x)e^{-xt}.$$

In summary we have been able to solve partial differential equations in the case in which they can be integrated. The arbitrary constants that appear are replaced by arbitrary functions of the "other" independent variable.

## EXERCISES

65.1. Determine the solution of $\partial\rho/\partial t = (\sin x)\rho$ which satisfies $\rho(x, 0) = \cos x$.

65.2. Determine the solution of $\partial\rho/\partial t = \rho^2$ which satisfies $\rho(x, 0) = \sin x$.

65.3. Determine the solution of $\partial\rho/\partial t = \rho$, which satisfies $\rho(x, t) = 1 + \sin x$ along $x = -2t$.

65.4. Is there a solution of $\partial\rho/\partial t = -x^2\rho$, such that both $\rho(x, 0) = \cos x$ for $x > 0$ and $\rho(0, t) = \cos t$ for $t > 0$?

65.5. Determine the solution of $\partial\rho/\partial t = xt\rho$ which satisfies $\rho(x, 0) = f(x)$.

# 66. *Linearization*

The partial differential equation which was formulated to mathematically model traffic flow is

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho u(\rho)) = 0 \tag{66.1a}$$

or equivalently

$$\frac{\partial\rho}{\partial t} + \frac{dq}{d\rho}\frac{\partial\rho}{\partial x} = 0. \tag{66.1b}$$

One possible initial condition is to prescribe the initial traffic density

$$\rho(x, 0) = f(x).$$

We will solve this problem, that is determine the traffic density at all future times.

This partial differential equation cannot be directly integrated as could the simple examples in the previous section, since both $\partial\rho/\partial t$ and $\partial\rho/\partial x$ appear in the equation. Although we will be able to solve this partial differ-